Unit 11

Fourier series

Introduction

This unit is concerned with *periodic functions*. A simple example is provided by the cosine function shown in Figure 1.

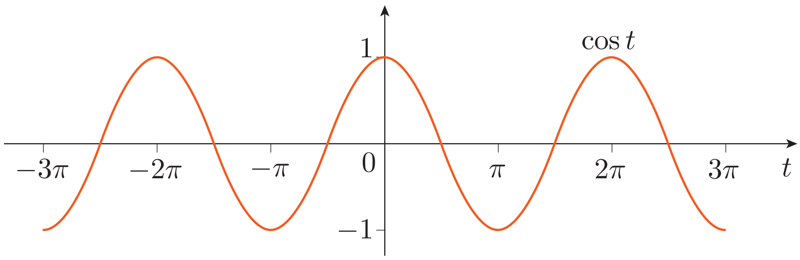


Figure 1 The cosine function $\cos t$, which has period 2π

This is a periodic function that repeats itself over every 2π interval of its domain. That is, if we plot the function for $0 \le t \le 2\pi$, and translate that piece of the function by $\pm 2\pi, \pm 4\pi, \ldots$ along the horizontal t-axis, then we recover the entire function. The same is true, of course, if we plot the function over any other domain interval of length 2π. A more mathematical way of expressing this fact is to say that $\cos t$ and $\cos(t + 2\pi)$ have the same value for all t. We write

$$\cos(t + 2\pi) = \cos(t) \quad \text{for all } t,$$

and say that $\cos t$ has *period* 2π.

Figures 2–4 show three more examples of periodic functions.

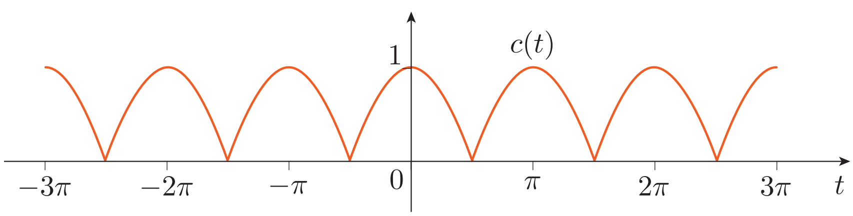


Figure 2 The function $c(t) = |\cos t|$, with period π

The function $c(t)$ shown in Figure 2 is the modulus of the cosine function, defined by

$$c(t) = |\cos t|.$$

This function repeats itself every domain interval of length π. Its period is π, and we write

$$c(t + \pi) = c(t) \quad \text{for all } t.$$

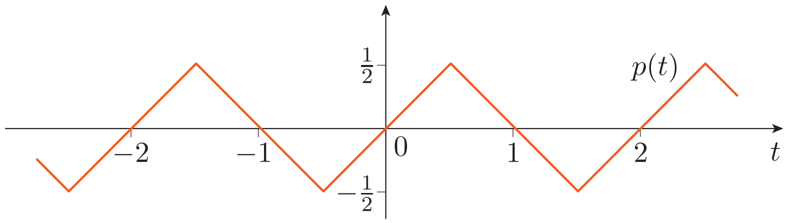


Figure 3 A sawtooth function $p(t)$, with period 2

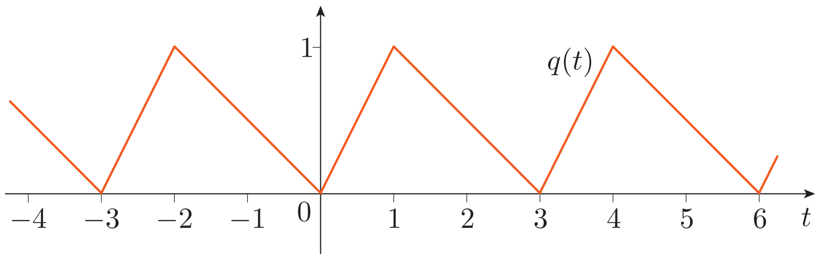


Figure 4 A sawtooth function $q(t)$, with period 3

The functions $p(t)$ and $q(t)$ shown in Figures 3 and 4 are of a type known as **sawtooth functions** because their graphs are similar in shape to the teeth of a saw. There are no simple one-line formulas for functions like this, but you will see how they can be specified in Section 1. The graph of $p(t)$ in Figure 3 repeats itself over every domain interval of length 2. Its period is 2, and we write

$$p(t+2) = p(t) \quad \text{for all } t.$$

The graph of $q(t)$ in Figure 4 repeats itself over every domain interval of length 3. Its period is 3, and we write

$$q(t+3) = q(t) \quad \text{for all } t.$$

Periodicity and hence periodic functions arise naturally in a wide variety of contexts. They may describe a regular oscillation in time or a regular variation in space. For the most part we will ignore the physical setting and just think about periodic functions of an independent variable.

The importance of periodic functions

A familiar example of a periodic function arises when an undamped pendulum or an undamped harmonic oscillator (of the type discussed in Unit 3) moves to and fro; in this case, the displacement is a periodic function of time. Many musical instruments are mechanical oscillators: for example, you can see the vibrating motion of the strings of a guitar or a piano. Other musical instruments, such as an organ or a flute, create oscillations of pressure in the air.

Periodic functions are often a consequence of circular or cyclic motion. For example, the number of hours of daylight in London varies periodically with a period of one year (with a maximum

around 21 June and a minimum around 21 December). This periodic variation is a consequence of the orbit of the Earth around the Sun. Other examples of approximately periodic functions of time, such as the rise and fall of the tides, can be traced to the orbits of astronomical bodies. (In the case of tides, the effect is predominantly due to the gravitational influence of the Moon.)

Many mechanical systems naturally give rise to periodic functions of time. An example is the steam locomotive shown in Figure 5, where the periodic motion of a piston inside a cylinder drives the rotation of a set of wheels through a connecting rod. A very similar arrangement for converting the periodic motion of a piston to circular motion is used in almost all car engines.

In science and engineering, we encounter periodic functions of position as well as time. A very important example is the arrangement of atoms in a crystal, where the atoms form a regular arrangement on a three-dimensional lattice, such as that illustrated in Figure 6. Here the density of electrons in the crystal is described by a periodic function of three spatial variables. However, this is beyond the scope of this unit, which considers only periodic functions of a single variable.

Figure 5 Periodic and cyclical motions are often closely related: for example, periodic motion of a piston in a cylinder, driven by escaping steam, is converted to a circular motion of a set of wheels on this locomotive

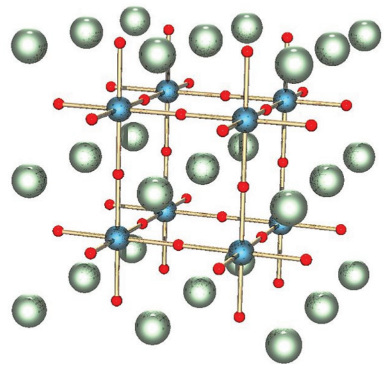


Figure 6 The atoms of many materials, such as this perovskite crystal, have a regular and periodic spacing; the electron density in such a crystal is a periodic function of position

Obviously sines and cosines are periodic functions. However, it turns out that (almost) all periodic functions can be written in a unified way: as a constant (which may be zero) plus an infinite sum over sines and cosines. In this way sines and cosines can be viewed as the 'fundamental' periodic functions, i.e. all others can be expressed as linear combinations of them.

For example, it can be shown that the function $c(t) = |\cos t|$ corresponds to the infinite sum

$$C(t) = \frac{4}{\pi}\left[\tfrac{1}{2} + \tfrac{1}{3}\cos(2t) - \tfrac{1}{15}\cos(4t) + \tfrac{1}{35}\cos(6t) - \cdots\right]. \qquad (1)$$

This sum contains infinitely many terms. In the limiting case where all these terms are added together, the sum $C(t)$ becomes equal to the original function $c(t)$, allowing us to write

$$c(t) = C(t).$$

In a similar way, the sawtooth functions $p(t)$ and $q(t)$ introduced above correspond to the infinite sums

$$P(t) = \frac{4}{\pi^2}\left[\sin(\pi t) - \tfrac{1}{9}\sin(3\pi t) + \tfrac{1}{25}\sin(5\pi t) - \tfrac{1}{49}\sin(7\pi t) + \cdots\right], \qquad (2)$$

$$\begin{aligned} Q(t) = \frac{1}{2} &- \frac{27}{8\pi^2}\left[\cos\left(\tfrac{2\pi t}{3}\right) + \tfrac{1}{4}\cos\left(\tfrac{4\pi t}{3}\right) + \tfrac{1}{16}\cos\left(\tfrac{8\pi t}{3}\right) + \cdots\right] \\ &+ \frac{9\sqrt{3}}{8\pi^2}\left[\sin\left(\tfrac{2\pi t}{3}\right) - \tfrac{1}{4}\sin\left(\tfrac{4\pi t}{3}\right) + \tfrac{1}{16}\sin\left(\tfrac{8\pi t}{3}\right) + \cdots\right]. \end{aligned} \qquad (3)$$

When all the terms in these sums are added together, we get $p(t) = P(t)$ and $q(t) = Q(t)$.

Infinite sums like these are called **Fourier series**, named after their discoverer Joseph Fourier. They have a common structure, which can be written as

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega_n t) + \sum_{n=1}^{\infty} B_n \sin(\omega_n t),$$

where A_0, A_n, B_n and ω_n are real numbers, some of which may be equal to zero (details will be given in Subsection 2.1).

Notice that we distinguish the Fourier series from its corresponding function by using a capital letter, so that the Fourier series for $c(t)$ is denoted $C(t)$. For all the functions that we will consider, the function $f(t)$ and its Fourier series $F(t)$ have the same values, i.e. $f(t) = F(t)$ (except perhaps at isolated points), even though they may be completely different expressions.

There are some interesting differences between the Fourier series in equations (1)–(3). $C(t)$ is a constant plus a sum over cosines, $P(t)$ is a sum over sines, and $Q(t)$ is a constant plus a sum over both sines and cosines.

Since sines and cosines are periodic, it is perhaps not too surprising that $C(t)$, $P(t)$ and $Q(t)$ are also periodic. Note also that for successive terms in each sum, the argument of the sine or cosine increases, but the constant multiplying the term decreases. The sums are infinite in the sense that they do not stop after a finite number of terms, although in practice we take only enough terms to make the result as accurate as required. In this unit you will see how to calculate Fourier series. This means finding appropriate values for A_0, A_n, B_n and ω_n for any given periodic function.

To see how it is possible for a sum over sines and cosines to approach a given function, take a look at Figure 7.

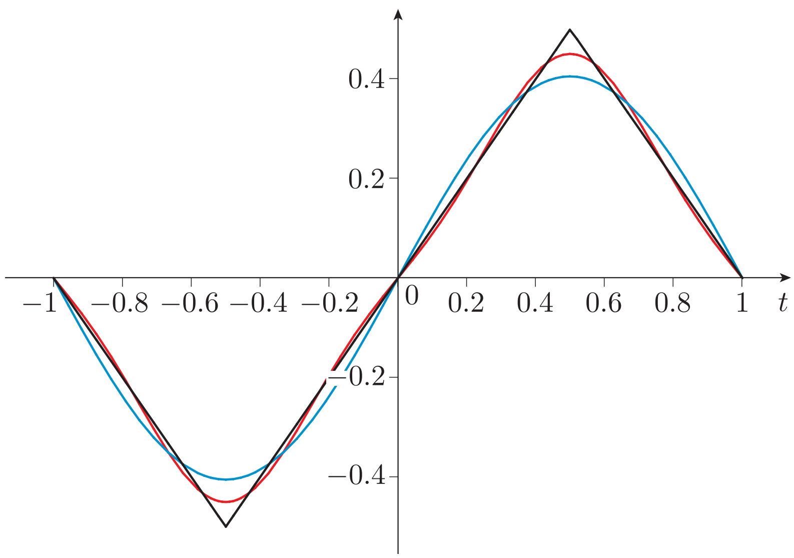


Figure 7 The function $p(t)$ (in black), together with the approximations $P_1(t)$ (in blue) and $P_2(t)$ (in red) representing the first term and the sum of the first two terms in equation (2), the Fourier series for $p(t)$

Here we compare the exact sawtooth function $p(t)$ of Figure 3 with the first two approximations to its Fourier series $P(t)$. The first approximation $P_1(t)$ is just the first term in $P(t)$, and the second approximation $P_2(t)$ is the sum of the first two terms:

$$P_1(t) = \frac{4}{\pi^2}\sin(\pi t),$$
$$P_2(t) = \frac{4}{\pi^2}\left[\sin(\pi t) - \tfrac{1}{9}\sin(3\pi t)\right].$$

You can see that P_2 (in red) is a better approximation than P_1 (in blue). More generally, as we add more terms to the Fourier series, it gradually approaches the original function. Each additional term improves the approximation. In fact, if we were to plot $P_{10}(t)$, the sum of the first 10 terms in the Fourier series, it would be hardly distinguishable from the original function. Adding all the terms in the Fourier series would give $p(t) = P(t)$ exactly.

Figure 8 The French mathematician, physicist and historian Joseph Fourier (1768–1830)

Fourier series will be used in Unit 12. There you will be looking at the transverse vibrations of guitar strings and the conduction of heat along metal rods. In the case of a vibrating guitar string, it is not surprising that periodic functions are involved and that Fourier series are applicable. In fact, by describing the shape of a displaced guitar string by a Fourier series, we can represent the sound made by a guitar as a sum of a fundamental tone and a series of harmonics. However, it is not so obvious that solutions to the heat conduction problem can also be found as sums of sinusoidal terms. This was one of the many great discoveries of Joseph Fourier (Figure 8).

Study guide

This unit shows you how to calculate the Fourier series for periodic and other functions, like those in equations (1)–(3). It assumes that you are familiar with *integration by parts* and *complex numbers* (see Unit 1).

Section 1 defines what periodic functions are and explains how to write down their formulas. Much of this may be familiar to you, but we recommend that you read it and attempt the exercises as the ideas are used throughout the rest of the unit.

Section 2 is the core section of this unit. It defines the Fourier series for a periodic function, and shows how such a series is calculated. It also explains how Fourier series can be used to represent non-periodic functions defined over a finite domain. Section 3 discusses the behaviour of Fourier series near discontinuities, and considers the differentiation of Fourier series.

The remainder of the unit consists of Section 4 and an Appendix. These parts of the unit are optional, but we strongly advise you to study them if you have time, because they develop ideas that are important in more advanced areas of mathematics and the physical sciences. Section 4 introduces an alternative form of the Fourier series, based on a sum of complex exponential functions. We illustrate the value of this form of the

Fourier series by showing how it can help us to solve a differential equation. Finally, the Appendix contains proofs of formulas used in the main text, and also develops the important idea of orthogonal functions.

1 Periodic functions

Before explaining how to calculate Fourier series, we first look at the definition and properties of periodic functions. We then briefly review the types of notation that are used to express periodic functions. Finally, we remind you about some properties of odd and even functions. Much of this material may already be familiar to you, but these subsections are short and the material contained in them is essential to what follows. So it is worth reading them and trying the exercises before moving on to Section 2.

1.1 The period of a function

In the Introduction we looked at some examples of periodic functions, and stated that they have the property that, when plotted, they *repeat themselves* over some interval. We now make this intuitive notion of periodicity precise.

Consider the function $\cos t$ shown in Figure 9.

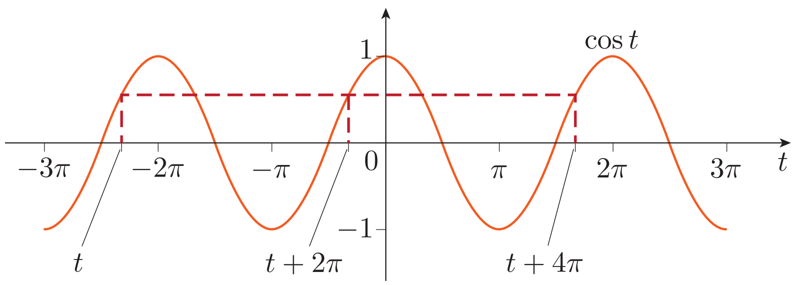


Figure 9 The periodicity of the cosine function: $\cos t = \cos(t + 2\pi) = \cos(t + 4\pi) = \cos(t + 6\pi) = \dots$

Notice that $\cos 0 = \cos 2\pi$. In fact, it is clear that $\cos t = \cos(t + 2\pi)$ for all values of t. Likewise, we can say that

$$\cos t = \cos(t + 2\pi) = \cos(t + 4\pi) = \cos(t + 6\pi) = \dots \quad \text{for all } t.$$

So there is an infinite set of positive values $\tau = 2\pi, 4\pi, 6\pi, \dots$ for which

$$\cos t = \cos(t + \tau) \quad \text{for all } t.$$

We call these values of τ the *periods* of $\cos t$. The smallest (non-zero) period is $\tau = 2\pi$, and this is called the *fundamental period* of $\cos t$. More generally, we make the following definitions.

Periodic functions and their periods

A function $f(t)$ is said to be **periodic** if for some *positive* number τ it satisfies

$$f(t) = f(t + \tau) \quad \text{for all } t.$$

The number τ is said to be a **period** of the function $f(t)$.

If τ is a period of $f(t)$, then so are 2τ, 3τ, and so on. By definition, all periods are positive, and the smallest period of $f(t)$ is called the **fundamental period**.

In applications, the fundamental period is far more important than the other periods. For this reason, many scientists use the term *period* as a shorthand for the fundamental period. For example, the fundamental period of a pendulum (the time it takes to swing to and fro) is usually called *the* period of the pendulum. We occasionally use this shorthand when there is no risk of confusion. If we talk about *the period* of a function, then we mean its fundamental period, but if we talk of the *set of periods* of a function, then we mean all of its periods, fundamental and not.

We sometimes need to find the period of a function such as $\cos(5t)$. In this case, we can argue as follows. The complete set of periods of the cosine function is $\tau = 2\pi, 4\pi, 6\pi, 8\pi, \ldots$. For any of these values of τ, we have

$$\cos(5(t + \tau/5)) = \cos(5t + \tau) = \cos(5t).$$

This shows that $\tau/5$ is one of the periods of $\cos(5t)$. The smallest such period corresponds to the smallest of the above values of τ, which is 2π, so the fundamental period of $\cos(5t)$ is $2\pi/5$. More generally, the constant ω in the functions $\cos(\omega t)$ and $\sin(\omega t)$ is called the **angular frequency**, and the fundamental period of these functions is $2\pi/\omega$.

Example 1

What are the fundamental periods of the following functions?

(a) $\sin 5t$

(b) $\cos 3t$

(c) $3 \sin 5t + 7 \cos 3t$

Solution

(a) The function $\sin 5t$ has angular frequency $\omega = 5$ and therefore has fundamental period $\tau = 2\pi/\omega = 2\pi/5$.

(b) The function $\cos 3t$ has angular frequency $\omega = 3$ and therefore has fundamental period $\tau = 2\pi/\omega = 2\pi/3$.

(c) The function $3\sin 5t + 7\cos 3t$ is the sum of two functions: $3\sin 5t$ and $7\cos 3t$. The complete set of periods for $3\sin 5t$ is given by the positive integer multiples of $2\pi/5$, that is,

$$\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \frac{10\pi}{5}, \ldots .$$

The complete set of periods for $7\cos 3t$ is given by the positive integer multiples of $2\pi/3$, that is,

$$\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \ldots .$$

The smallest period that these functions have in common is $10\pi/5 = 2\pi$ for $3\sin 5t$ and $6\pi/3 = 2\pi$ for $7\cos 3t$. So the fundamental period of $3\sin 5t + 7\cos 3t$ is 2π.

Example 2

Find $p(99)$ and $p(99.5)$ for the sawtooth function $p(t)$ in Figure 3.

Solution

The given function has period 2, so $p(t+2) = p(t)$. Hence

$$p(99) = p(97) = p(95) = \ldots = p(1) = 0,$$

where the final value is obtained by examination of Figure 3. Similarly,

$$p(99.5) = p(97.5) = p(95.5) = \ldots = p(1.5) = -\tfrac{1}{2}.$$

Exercise 1

What are the fundamental periods of the following functions?

(a) $\sin\left(\frac{1}{4}x\right)$ (b) $\cos\left(\frac{2}{5}x\right)$ (c) $\sin\left(\frac{1}{4}x\right) + 2\cos\left(\frac{2}{5}x\right)$

Exercise 2

Calculate $q(1000)$ and $q(-77)$, where $q(t)$ is the sawtooth function in Figure 4.

1.2 Piecewise functions

We have previously noted that the two sawtooth functions $p(t)$ and $q(t)$, sketched in Figures 3 and 4, cannot be described by simple one-line formulas. However, we can write down relatively simple formulas for these functions if we initially restrict attention to a finite interval and split this into two pieces.

First consider the function $p(t)$, whose graph is reproduced in Figure 10.

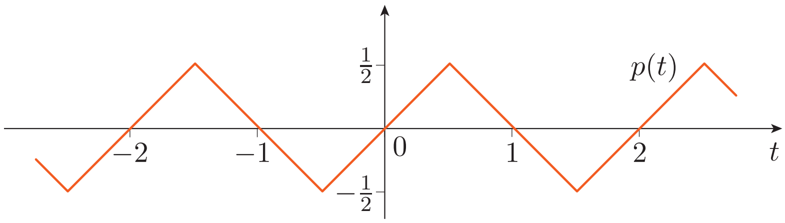


Figure 10 The function $p(t)$

Notice that if t lies in the interval $-\frac{1}{2} \leq t \leq \frac{1}{2}$, then $p(t) = t$. Also, if t lies in the interval $\frac{1}{2} \leq t \leq \frac{3}{2}$, then $p(t) = 1 - t$. This allows us to use the following notation to define $p(t)$ on the interval $-\frac{1}{2} \leq t \leq \frac{3}{2}$:

$$p(t) = \begin{cases} t & \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2}, \\ 1 - t & \text{for } \frac{1}{2} < t \leq \frac{3}{2}. \end{cases}$$

In this expression, we have chosen to include the point $t = \frac{1}{2}$ in the first region, which is written as $-\frac{1}{2} \leq t \leq \frac{1}{2}$. There is then no need to include $t = \frac{1}{2}$ in the second region, so this is written as $\frac{1}{2} < t \leq \frac{3}{2}$.

Different choices are equally valid. For example, the two regions could be taken to be $-\frac{1}{2} \leq t < \frac{1}{2}$ and $\frac{1}{2} \leq t \leq \frac{3}{2}$.

Functions like this, which are defined on two or more pieces of their domain, are called **piecewise functions**. We now complete the definition of $p(t)$ for all t by adding the information that it is periodic with period 2, i.e. $p(t + 2) = p(t)$. So the full definition of $p(t)$ is

$$\begin{aligned} &p(t) = \begin{cases} t & \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2}, \\ 1 - t & \text{for } \frac{1}{2} < t \leq \frac{3}{2}, \end{cases} \\ &p(t + 2) = p(t). \end{aligned} \tag{4}$$

The condition $p(t + 2) = p(t)$ applies for all t.

The first part of this definition defines $p(t)$ on an interval of length $\tau = 2$. This interval is called the **fundamental interval** of the function. The second part tells us that the function has fundamental period $\tau = 2$.

Note that because $p(t)$ is periodic, it is quite permissible to define it on any other fundamental interval of length 2. For example,

$$\begin{aligned} &p(t) = \begin{cases} t & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 1 - t & \text{for } \frac{1}{2} < t \leq \frac{3}{2}, \\ t - 2 & \text{for } \frac{3}{2} < t \leq 2, \end{cases} \\ &p(t + 2) = p(t) \end{aligned}$$

is an equally valid, if less elegant, definition. In this case the fundamental interval is $0 \leq t \leq 2$, while in equation (4) the fundamental interval is $-\frac{1}{2} \leq t \leq \frac{3}{2}$.

Example 3

Give a piecewise definition of the function $q(t)$ in Figure 11, using the fundamental interval $0 \leq t \leq 3$.

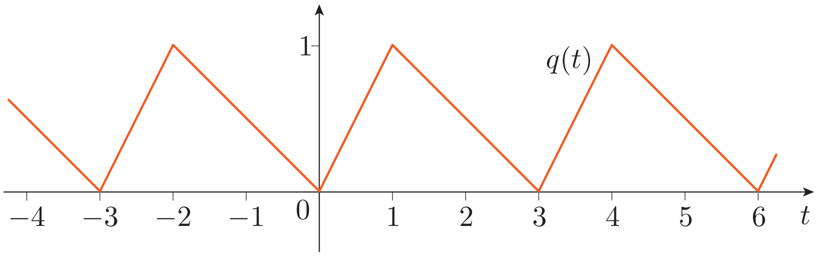


Figure 11 The function $q(t)$

Solution

We have $q(t) = t$ for $0 \leq t \leq 1$, and $q(t) = \frac{3}{2} - \frac{1}{2}t$ for $1 < t \leq 3$. Furthermore, $q(t)$ has period 3, so $q(t+3) = q(t)$ for all t. We can therefore define $q(t)$ as follows:

$$q(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \leq 3, \end{cases}$$
$$q(t+3) = q(t).$$

Exercise 3

Write down the piecewise definition of the function $q(t)$ in Figure 11 over the fundamental interval $-\frac{3}{2} \leq t \leq \frac{3}{2}$.

Exercise 4

Sketch the following function over the range $-3 \leq x \leq 6$:

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ 1 & \text{for } 1 < x \leq 2, \\ 3 - x & \text{for } 2 < x \leq 3, \end{cases}$$
$$f(x+3) = f(x).$$

1.3 Discontinuous functions

The function $\cos t$ shown in Figure 1 is both **continuous** and **smooth**. A *continuous* function can be sketched without lifting your pen from the paper – there are no abrupt changes in value. A *smooth* function has a graph with no sharp corners – there are no abrupt changes in slope. The functions $c(t)$, $p(t)$ and $q(t)$ in Figures 2–4 are all continuous, but they are not smooth because these graphs have sharp corners. In this subsection, we introduce functions that are not continuous.

Consider the function $h(t)$ shown in Figure 12.

The dashed lines are used to guide the eye; they are not part of the definition of the function.

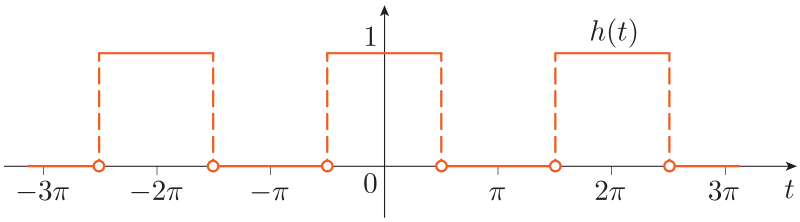


Figure 12 A square-wave function $h(t)$ with period 2π

This is a type of function known as a **square-wave function**, and the version considered here is defined by

$$h(t) = \begin{cases} 1 & \text{for } -\frac{1}{2}\pi \leq t \leq \frac{1}{2}\pi, \\ 0 & \text{for } \frac{1}{2}\pi < t < \frac{3}{2}\pi, \end{cases} \tag{5}$$

$$h(t + 2\pi) = h(t).$$

The period of this function is 2π, and the fundamental interval used in the above definition is $-\frac{1}{2}\pi \leq t < \frac{3}{2}\pi$. This function is *not* continuous since $h(t)$ has value 1 or 0 and there are abrupt changes in the function value at the points $t = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$. Such functions are said to be **discontinuous** or **non-continuous**.

As noted earlier, we sometimes use period to mean fundamental period.

In Figure 12, $h(t)$ is drawn using a convention that is sometimes used to specify a function at points of discontinuity. Intervals of the form $\frac{1}{2}\pi < t < \frac{3}{2}\pi$ are drawn with small open circles at the ends to denote the missing endpoints.

This convention is used in mathematics, but most science texts do not bother with it.

We need to be a little careful about piecewise function definitions when dealing with discontinuous functions. Remember that functions can take only *one* value for each point in their domain. That is why we have not included the point $t = \frac{1}{2}\pi$ in both halves of the function definition (5). In the first line of this definition, we have chosen to take $h = 1$ at $t = \frac{1}{2}\pi$; it would not be consistent to assign the value $h = 0$ at $t = \frac{1}{2}\pi$ in the second line of the definition. Also, we do not define the function value 0 at $t = \frac{3}{2}\pi$ because that would contradict the first line and the property of periodicity, which taken together give

$$h\left(\tfrac{3}{2}\pi\right) = h\left(-\tfrac{1}{2}\pi + 2\pi\right) = h\left(-\tfrac{1}{2}\pi\right) = 1.$$

We could equally well define a slightly different square-wave function $h(t)$, taking the value 0 at the points $\pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$. This would then be written as

Strictly speaking, we should use a different symbol for this function, but the change is so minor that we do not do so.

$$h(t) = \begin{cases} 1 & \text{for } -\frac{1}{2}\pi < t < \frac{1}{2}\pi, \\ 0 & \text{for } \frac{1}{2}\pi \leq t \leq \frac{3}{2}\pi, \end{cases} \tag{6}$$

$$h(t + 2\pi) = h(t),$$

and drawn as shown in Figure 13.

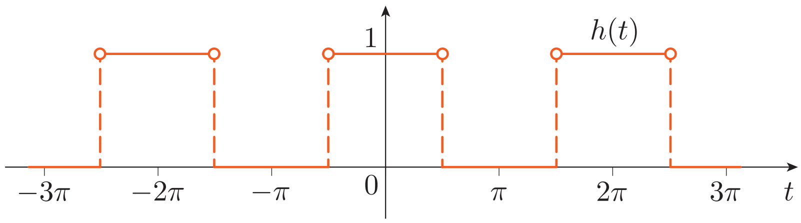


Figure 13 A slightly different definition of $h(t)$

You should check that this definition is internally consistent and does not attempt to give two different values at the same point.

Discontinuities in graphs

The real world almost always deals with continuous functions. In fact, discontinuous functions normally arise in science as convenient approximations to continuous functions. For this reason, scientists often take a relaxed attitude to the definitions and graphs of discontinuous functions, omitting the small open circles at points of discontinuity. We are more careful, but in the end this detail makes no *physical* difference.

You will see that discontinuous functions have Fourier series. For example, it turns out that the Fourier series for $h(t)$ is given by

$$H(t) = \frac{1}{2} + \frac{2}{\pi}\cos t - \frac{2}{3\pi}\cos 3t + \frac{2}{5\pi}\cos 5t - \frac{2}{7\pi}\cos 7t + \cdots . \qquad (7)$$

This Fourier series applies whether $h(t)$ is defined by equation (5) or equation (6). This is because Fourier series *do not distinguish between functions that differ only at isolated points*. It is quite a remarkable result that a discontinuous function can be written as a sum of continuous and smooth functions, i.e. sines and cosines. We will have more to say about the Fourier series for discontinuous functions in Subsection 3.1.

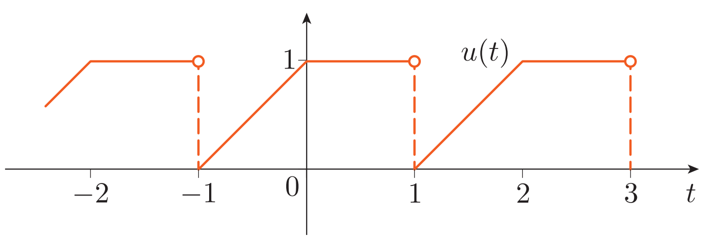
Exercise 5

Sketch the following function over $-3\pi \le t < 3\pi$:

$$f(t) = t \quad \text{for } -\pi \le t < \pi,$$
$$f(t + 2\pi) = f(t).$$

Exercise 6

Consider the function $u(t)$ shown in the figure below.



(a) State the positions of any discontinuities in the function, and give the value of its fundamental period.

(b) Write down the piecewise definition of the function, using the fundamental interval $-1 \leq t < 1$.

(c) What are the values of $u(99)$ and $u(100)$?

1.4 Even and odd functions

In the next section we will calculate Fourier series for periodic functions. In addition to being periodic, some of these functions will be even or odd, and this can simplify the calculations. It is therefore useful to review some properties of even and odd functions that were discussed in Unit 1. Recall that even and odd functions are defined as follows.

Definitions

A function $f(t)$ is said to be **even** if $f(-t) = f(t)$ for all t.

A function $f(t)$ is said to be **odd** if $f(-t) = -f(t)$ for all t.

Most functions are neither even nor odd.

Typical examples of even functions are 1, x^2, $2x^2 + 4x^4$, $\cos x$ and $\exp(x^2)$.

Typical examples of odd functions are x, $x + 4x^5$, $\sin x$ and $x \exp(x^2)$.

It is easy to recognise even and odd functions from their graphs. An even function, such as that shown in Figure 14, takes the same values at corresponding points on either side of the vertical axis, so its graph remains unchanged when it is reflected in the vertical axis.

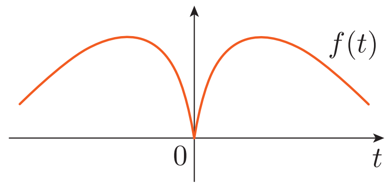


Figure 14 An even function

By contrast an odd function, such as that shown in Figure 15, has values of the same magnitude but opposite signs at corresponding points on either side of the vertical axis, so its graph changes sign when it is reflected in the vertical axis.

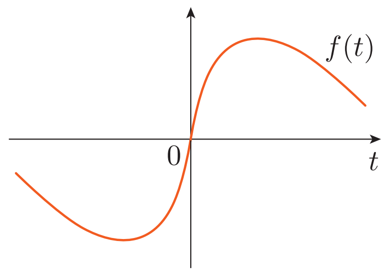


Figure 15 An odd function

Returning to the functions considered at the beginning of the Introduction, we see that those in Figures 1 and 2 are even, while that in Figure 3 is odd. The function in Figure 4 is neither even nor odd.

When we calculate Fourier series, we will need to evaluate definite integrals. It is useful to note that simplifications occur when even and odd functions are integrated over a range that is symmetric about the origin.

Recall that if $f(t)$ is an odd function, then

$$\int_{-a}^{a} f(t)\, dt = 0.$$

The reason, illustrated in Figure 16, is that the area under $f(t)$ between $-a$ and 0 (in yellow) has the same magnitude but opposite sign to the area between 0 and a (in blue). So the total area between $-a$ and a vanishes.

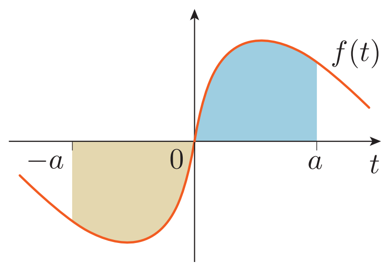


Figure 16 The integral of an odd function $f(t)$ over a range from $-a$ to a

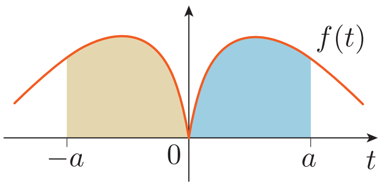


Figure 17 The integral of an even function $f(t)$ over a range from $-a$ to a

Now consider the integral of an even function $f(t)$ between $-a$ and a, as illustrated in Figure 17. Clearly the area under the function between $-a$ and a is just twice that between 0 and a. This means that

$$\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$$

for any even function $f(t)$.

We collect these results for later use.

Integrals of odd and even functions over symmetric ranges

If $f(x)$ is an *odd* function, then

$$\int_{-a}^{a} f(t)\,dt = 0. \tag{8}$$

If $f(x)$ is an *even* function, then

$$\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt. \tag{9}$$

Example 4

Calculate the value of

$$I = \int_{-1}^{1} \sin(x^3)\cos(x^2)\,dx.$$

Solution

The integrand is $f(x) = \sin(x^3)\cos(x^2)$. This is odd because

$$\begin{aligned} f(-x) &= \sin((-x)^3)\cos((-x)^2) \\ &= \sin(-x^3)\cos(x^2) \\ &= -\sin(x^3)\cos(x^2) = -f(x). \end{aligned}$$

The range of integration is symmetric about the origin, so

$$I = \int_{-1}^{1} \sin(x^3)\cos(x^2)\,dx = 0.$$

In general, the product of an odd function and an even function is odd, and the product of two odd functions or two even functions is even. Also, the sum of two odd functions is odd, and the sum of two even functions is even.

Exercise 7

Determine whether the following functions are even, odd or neither.

(a) $x^3 - 3x$ (b) $2\sin x + 3\sin(4x)$ (c) $5 + 2\cos x + 7\cos(4x)$

(d) $4 - 2\sin x$ (e) $2x\cos(3x)$

Exercise 8

Calculate the values of the following definite integrals.

(a) $\displaystyle\int_{-1}^{1} x^3\cos(2x)\,dx$ (b) $\displaystyle\int_{-2}^{2}(3+\sin(2x^3))\,dx$

2 Introducing Fourier series

This section contains the core material of this unit. It defines what is meant by a Fourier series and shows you the basic method for calculating the Fourier series for any periodic function. The process of finding the Fourier series for a function is given in Procedure 1. The three examples that follow apply this procedure to specific periodic functions. Subsection 2.2 shows how this procedure can be simplified if the periodic function is either even or odd. Finally, Subsection 2.3 shows how Fourier series can be calculated for non-periodic functions that are defined over a finite domain.

2.1 Fourier series for periodic functions

Suppose that we have a periodic function $f(t)$ with fundamental period τ. Then we *define* its Fourier series to have the form

$$\begin{aligned}F(t) = A_0 &+ \left(A_1\cos\left(\frac{2\pi t}{\tau}\right) + A_2\cos\left(\frac{4\pi t}{\tau}\right) + A_3\cos\left(\frac{6\pi t}{\tau}\right) + \cdots\right)\\ &+ \left(B_1\sin\left(\frac{2\pi t}{\tau}\right) + B_2\sin\left(\frac{4\pi t}{\tau}\right) + B_3\sin\left(\frac{6\pi t}{\tau}\right) + \cdots\right),\end{aligned}$$

where the sums may continue forever. More concisely, the Fourier series can be written as

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{2n\pi t}{\tau}\right), \tag{10}$$

or as

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos(\omega_n t) + \sum_{n=1}^{\infty} B_n\sin(\omega_n t), \tag{11}$$

where

$$\omega_n = \frac{2n\pi}{\tau} \quad (n = 1, 2, 3, \ldots).$$

Each periodic function $f(t)$ of fundamental period τ has its own Fourier series $F(t)$, characterised by a particular set of constants A_0, A_n, B_n $(n = 1, 2, 3, \ldots)$. These are called the **Fourier coefficients** for $f(t)$.

Finding the Fourier series for a function means finding its Fourier coefficients A_0, A_n, B_n for $n = 1, 2, 3, \ldots$.

As stated in the Introduction, a periodic function $f(t)$ and its Fourier series $F(t)$ are the same function, i.e. $f(t) = F(t)$ (except perhaps at isolated points), even though their formulas look quite different.

Since $f(t)$ has period τ, we know that $f(t+\tau) = f(t)$ for all t. Let us check that this property is also true for the Fourier series $F(t)$. We can do this by noting that $\cos(\omega t)$ and $\sin(\omega t)$ have angular frequency ω and fundamental period $\tau = 2\pi/\omega$. Similarly, $\cos(\omega_n t)$ and $\sin(\omega_n t)$ have angular frequency ω_n and fundamental period $2\pi/\omega_n = \tau/n$. Other periods of these functions are obtained by multiplying τ/n by any positive integer, so τ is one of the periods of $\cos(\omega_n t)$ and $\sin(\omega_n t)$. Since every term in the Fourier series of equation (11) has τ as one of its periods, we conclude that

$$F(t+\tau) = F(t) \quad \text{for all } t,$$

as required.

To illustrate the meaning of equations (10) and (11), let us return to the examples given in the Introduction. The function $c(t) = |\cos t|$, illustrated in Figure 2, has fundamental period $\tau = \pi$, so $\omega_n = 2n\pi/\pi = 2n$. We therefore expect this function to have a Fourier series of the form

$$C(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2nt) + \sum_{n=1}^{\infty} B_n \sin(2nt).$$

However, the Introduction stated that the Fourier series for $c(t)$ is given by equation (1). You can see that these two equations are equivalent if the Fourier coefficients for $c(t)$ are given by

$$A_0 = \frac{2}{\pi}, \quad A_1 = \frac{4}{3\pi}, \quad A_2 = -\frac{4}{15\pi}, \quad A_3 = \frac{4}{35\pi}, \quad \ldots,$$

with $B_1 = B_2 = B_3 = \ldots = 0$.

Exercise 9

Deduce the Fourier coefficients A_0, A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 for the sawtooth function $p(t)$ in Figure 3, whose Fourier series is given by equation (2).

The key question, of course, is how do you deduce the Fourier coefficients for a general periodic function $f(t)$? To begin to answer this question, let us suppose that a function $f(t)$ with fundamental period τ has Fourier series $F(t)$. The function and its Fourier series are supposed to have identical values, so we can write equation (10) as

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right). \tag{12}$$

We can get some useful information by integrating both sides of this equation with respect to t over a complete period of $f(t)$, from $t = -\tau/2$ to $t = \tau/2$.

The expression on the right-hand side of equation (12) is a sum of terms, and its integral is found by integrating these terms one by one. In other words, we assume that the summation and integral signs can be interchanged, giving

$$\int_{-\tau/2}^{\tau/2} f(t)\,dt = \int_{-\tau/2}^{\tau/2} A_0\,dt + \sum_{n=1}^{\infty} A_n \int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right) dt + \sum_{n=1}^{\infty} B_n \int_{-\tau/2}^{\tau/2} \sin\left(\frac{2n\pi t}{\tau}\right) dt. \qquad (13)$$

We can immediately see that the integrals of all the sine terms vanish. This is because $\sin(2n\pi t/\tau)$ is an odd function of t, and the range of integration is symmetric about the origin.

The integrals of the cosine terms also vanish, but for a different reason. In this case, we have

$$\begin{aligned}\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right) dt &= \left[\frac{\tau}{2n\pi} \sin\left(\frac{2n\pi t}{\tau}\right)\right]_{-\tau/2}^{\tau/2} \\ &= \frac{\tau}{2n\pi}\left(\sin(n\pi) - \sin(-n\pi)\right),\end{aligned}$$

For any integer m, $\sin(m\pi) = 0$.

and this is equal to zero for $n = 1, 2, 3, \ldots$.

The only term that survives on the right-hand side of equation (13) is the integral of A_0, which is given by

$$\int_{-\tau/2}^{\tau/2} A_0\,dt = \left[A_0 t\right]_{-\tau/2}^{\tau/2} = A_0\tau.$$

We therefore conclude that

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt. \qquad (14)$$

The coefficient A_0 in the Fourier series is therefore found by integrating $f(t)$ over its fundamental interval: this integral is the *average value* of $f(t)$.

Example 5

Use equation (14) to calculate the Fourier coefficient A_0 for the function $h(t)$ shown in Figure 13.

Solution

From Figure 13, it is clear that $h(t)$ has period $\tau = 2\pi$. Hence using equation (14) we have

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} h(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)\,dt.$$

This integral represents the area under the graph of $h(t)$ between $-\pi$ and π. An examination of the figure shows that this is equal to π. Hence

$$A_0 = \tfrac{1}{2},$$

in agreement with the stated Fourier series in equation (7).

Exercise 10

Use equation (14) to calculate the Fourier coefficient A_0 for the function $c(t) = |\cos t|$ shown in Figure 2.

The above calculation of A_0 relies on the fact that the integrals of sines and cosines appearing in equation (13) are all equal to zero. Without doing any calculations, you can get an idea of why this happens. It is because the sine and cosine functions in equation (13) oscillate, taking both positive and negative values. The integral from $t = -\tau/2$ to $t = \tau/2$ takes us over a whole number of periods of $\cos(2n\pi t/\tau)$ or $\sin(2n\pi t/\tau)$, and over any such range, the positive and negative contributions cancel, giving zero integrals. An example of this (with $n = 2$) is shown in Figure 18.

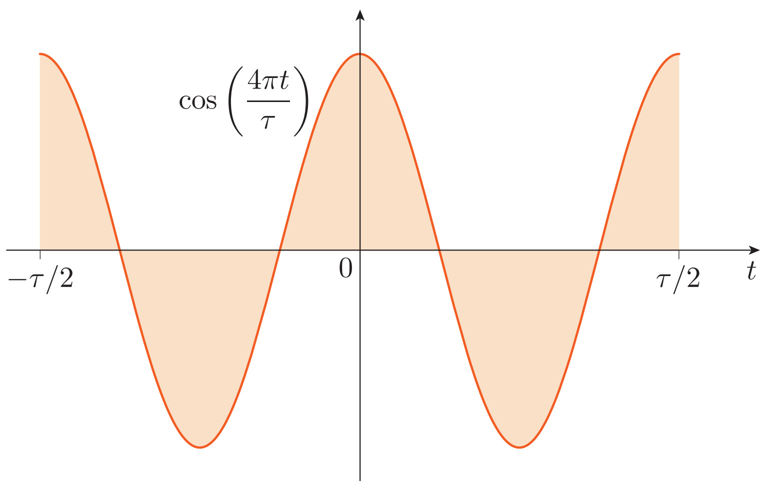


Figure 18 A graph of $\cos(4\pi t/\tau)$ from $-\tau/2$ to $\tau/2$

So far, we have found just one of the Fourier coefficients. The remaining coefficients can also be found by a process of integration. We will soon give formulas for them, and concentrate on the main task of calculating Fourier series for particular functions. The derivation of these formulas is given in the optional Appendix, but we give a sketch of the argument here, skimming over the fine details.

Suppose that we want to find the coefficient A_2. We note that the cosine that accompanies A_2 in the Fourier series is $\cos(4\pi t/\tau)$. The trick is then

to multiply both sides of equation (12) by $\cos(4\pi t/\tau)$ *before* integrating over t from $-\tau/2$ to $\tau/2$. This gives a modified version of equation (13), but with an extra factor $\cos(4\pi t/\tau)$ inside each integral.

Now cosines are always even functions, so $\cos(4\pi t/\tau)\sin(2n\pi t/\tau)$ is an odd function. It follows that

$$B_n \int_{-\tau/2}^{\tau/2} \cos\left(\frac{4\pi t}{\tau}\right) \sin\left(\frac{2n\pi t}{\tau}\right) dt = 0.$$

We also have

$$A_0 \int_{-\tau/2}^{\tau/2} \cos\left(\frac{4\pi t}{\tau}\right) dt = 0,$$

for the reasons outlined above – we are integrating an oscillating cosine function over a complete number of its periods, so the positive and negative contributions to the integral cancel. Hence our modified version of equation (13) simplifies to

$$\int_{-\tau/2}^{\tau/2} f(t) \cos\left(\frac{4\pi t}{\tau}\right) dt = \sum_{n=1}^{\infty} A_n \int_{-\tau/2}^{\tau/2} \cos\left(\frac{4\pi t}{\tau}\right) \cos\left(\frac{2n\pi t}{\tau}\right) dt. \quad (15)$$

The integrals on the right-hand side can be evaluated by using the trigonometric identity $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$, which converts a product of two cosines into a sum of two cosines. We can use this identity to write

$$\cos\left(\frac{4\pi t}{\tau}\right) \cos\left(\frac{2n\pi t}{\tau}\right) = \frac{1}{2}\left[\cos\left(\frac{(4+2n)\pi t}{\tau}\right) + \cos\left(\frac{(4-2n)\pi t}{\tau}\right)\right]. \quad (16)$$

We need to integrate this from $t = -\tau/2$ to $t = \tau/2$. Because n is an integer, we are integrating cosine functions over a whole number of their periods. As you have seen before, this means that the positive and negative contributions cancel, giving an integral that is equal to zero. This argument applies to almost every term on the right-hand side of equation (15), with just one exception. When $n = 2$, we have $4 - 2n = 0$ and one of the cosines in equation (16) is $\cos 0 = 1$. We therefore see that equation (15) reduces to

$$\int_{-\tau/2}^{\tau/2} f(t) \cos\left(\frac{4\pi t}{\tau}\right) dt = A_2 \int_{-\tau/2}^{\tau/2} \tfrac{1}{2}\, dt = \frac{\tau}{2} A_2,$$

which can be rearranged to give a formula for A_2:

$$A_2 = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos\left(\frac{4\pi t}{\tau}\right) dt.$$

With more work this method can be extended, providing us with formulas for all the Fourier coefficients. The complete set of these formulas is

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt,$$

$$A_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots),$$

$$B_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots).$$

Of course, we have only sketched the derivation of these results (further details can be found in the Appendix), but the really important thing is that these formulas exist and allow us to calculate the Fourier series for any periodic function $f(t)$.

Procedure 1 Fourier series for periodic functions

To find the Fourier series for a periodic function $f(t)$, proceed as follows.

1. Find the fundamental period τ.
2. Write down the Fourier series

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right), \tag{17}$$

   where A_0 and the A_n and B_n are coefficients to be determined. Simplify the arguments of the sines and cosines where possible.
3. Use the following formulas to determine the Fourier coefficients:

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt, \tag{18}$$

$$A_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots), \tag{19}$$

$$B_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots). \tag{20}$$

4. If desired, express the final Fourier series in a compact form with general formulas for its coefficients.

Because the Fourier series in equation (17) involves sines and cosines, we sometimes refer to it as the **trigonometric Fourier series**. This allows us distinguish it from the *exponential Fourier series*, which is discussed at the end of the unit. It is worth noting that the formula for A_0 contains the factor $1/\tau$, while the formulas for A_n and B_n contain the factor $2/\tau$.

Examples 6–8 below illustrate how to calculate the Fourier series for any periodic function. Sometimes the algebra gets a little tedious, and it is

important to work with care and patience, but if you persevere you will have understood the main idea of this unit. Later we will show you some tricks that can simplify the calculations. In the last step of the procedure, the following values are often helpful.

Some useful values

If n is any integer, then we have

$$\cos(n\pi) = (-1)^n, \tag{21}$$

$$\sin(n\pi) = 0, \tag{22}$$

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^{n/2} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd,} \end{cases} \tag{23}$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{for } n \text{ even,} \\ (-1)^{(n+3)/2} & \text{for } n \text{ odd.} \end{cases} \tag{24}$$

Example 6

Use Procedure 1 to find the Fourier series for the square-wave function

$$f(t) = \begin{cases} -1 & \text{for } -\frac{\pi}{2} < t < 0, \\ 1 & \text{for } 0 \le t \le \frac{\pi}{2}, \end{cases}$$

$$f(t+\pi) = f(t),$$

sketched in Figure 19.

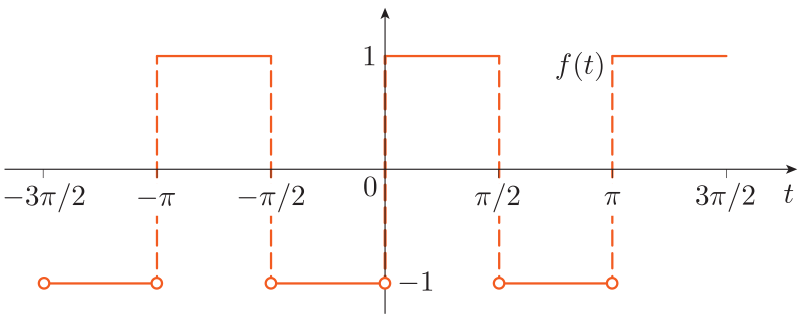


Figure 19 A particular type of square-wave function

Solution

The function $f(t)$ is odd and has fundamental period $\tau = \pi$. From equation (17), its Fourier series has the form

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2nt) + \sum_{n=1}^{\infty} B_n \sin(2nt), \tag{25}$$

where A_0, A_n and B_n $(n = 1, 2, 3, \dots)$ need to be determined.

We now calculate these Fourier coefficients. From equation (18) we get

$$A_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\,dt.$$

However, $f(t)$ is an odd function, and we know that the integral of an odd function over a range that is symmetric about the origin is equal to zero. Hence

$$A_0 = 0.$$

Using equation (19) we get

$$A_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\cos(2nt)\,dt.$$

Again, $f(t)\cos(2nt)$ is an odd function of t because $f(t)$ is odd and $\cos(2nt)$ is even, so

$$A_n = 0 \quad \text{for all } n = 1, 2, 3, \ldots.$$

Finally, from equation (20) we get

$$B_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(t)\sin(2nt)\,dt.$$

Since both $f(t)$ and $\sin(2nt)$ are odd, $f(t)\sin(2nt)$ is even, so using equation (9) we have

$$B_n = \frac{4}{\pi}\int_{0}^{\pi/2} f(t)\sin(2nt)\,dt.$$

But $f(t) = 1$ for $0 \le t \le \pi/2$, so

$$B_n = \frac{4}{\pi}\int_{0}^{\pi/2} \sin(2nt)\,dt = \frac{4}{\pi}\left[-\frac{1}{2n}\cos(2nt)\right]_0^{\pi/2} = -\frac{2}{n\pi}(\cos(n\pi) - 1),$$

where we have used $\cos 0 = 1$. Also, from equation (21) we have $\cos(n\pi) = (-1)^n$, so

$$B_n = \frac{2}{n\pi}(1 - (-1)^n).$$

Substituting the values for A_0, A_n and B_n into equation (25) gives

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - (-1)^n)\sin(2nt). \tag{26}$$

This is the Fourier series for the function $f(t)$ given in the question.

The first few non-zero terms of the Fourier series in equation (26) look like

$$\begin{aligned} F(t) &= \frac{2}{\pi}(1 - (-1)^1)\sin(2t) + \frac{2}{2\pi}(1 - (-1)^2)\sin(4t) \\ &\quad + \frac{2}{3\pi}(1 - (-1)^3)\sin(6t) + \cdots \\ &= \frac{4}{\pi}\sin(2t) + \frac{4}{3\pi}\sin(6t) + \frac{4}{5\pi}\sin(10t) + \cdots. \end{aligned}$$

To see how many terms are needed to get a good approximation to the original function $f(t)$, let us take the **truncated Fourier series** $F_N(t)$ to be the sum of the first N terms of the Fourier series. In this case,

$$F_N(t) = \sum_{n=1}^{N} \frac{2}{n\pi}(1-(-1)^n)\sin(2nt).$$

We would expect $F_N(t)$ to get closer and closer to $f(t)$ as N increases. This is borne out by Figure 20, which compares $F_5(t)$ (in blue) and $F_{20}(t)$ (in red) with the original function $f(t)$ (in black).

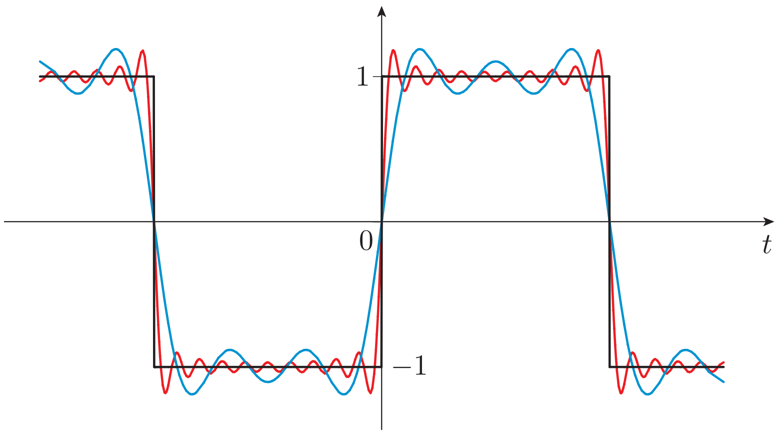


Figure 20 The square-wave function f (in black) together with its truncated Fourier series approximations F_5 (in blue) and F_{20} (in red)

In general, as the number of terms N increases, the truncated Fourier series $F_N(t)$ approaches the original function $f(t)$.

Fourier series can sometimes be written in alternative forms. Returning to equation (26), for example, we may notice that $1-(-1)^n = 0$ when n is even, and $1-(-1)^n = 2$ when n is odd. Hence the only values of n that contribute to this Fourier series are $n = 1, 3, 5, \ldots$. These values of n are given by $n = 2m-1$ for $m = 1, 2, 3, \ldots$, so the Fourier series also can be expressed as

$$F(t) = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\sin(2(2m-1)t). \tag{27}$$

Here m is just a label for different terms: you may replace it by n if you prefer.

Equations (26) and (27) are fully equivalent, and it doesn't matter which we use: both are the Fourier series for the function $f(t)$.

Let us pause for a moment to review what has been achieved. We assumed that the function $f(t)$ sketched in Figure 19 can be represented as a Fourier series of the form

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos(2nt) + \sum_{n=1}^{\infty} B_n\sin(2nt).$$

This means that $f(t)$ is equal to $F(t)$, except possibly at isolated points, so almost everywhere we can write

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2nt) + \sum_{n=1}^{\infty} B_n \sin(2nt), \tag{28}$$

where A_0, A_n and B_n for $n = 1, 2, 3, \ldots$ are constants that are initially unknown. The major task of finding these constants was carried out using equations (18)–(20). For example,

$$A_n = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \cos(2nt)\, dt \quad (n = 1, 2, 3, \ldots). \tag{29}$$

A deep analogy

The task of finding Fourier coefficients has much in common with the task of finding vector components. Suppose that we are given a vector $\mathbf{v}$ in three-dimensional space. We may know the magnitude and direction of this vector, but not its components. Nevertheless, we can write

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}, \tag{30}$$

where v_x, v_y and v_z are constants that are initially unknown, and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are Cartesian unit vectors. The problem of finding the constants is solved by taking scalar products with the unit vectors, giving

Equations like this were discussed in Unit 4.

$$v_x = \mathbf{i} \cdot \mathbf{v}, \quad v_y = \mathbf{j} \cdot \mathbf{v}, \quad v_z = \mathbf{k} \cdot \mathbf{v}. \tag{31}$$

Equations (28) and (30) are analogous because their right-hand sides are sums involving initially unknown constants. And equations (29) and (31) are analogous because they allow us to isolate individual constants from the sums. Most analogies are weak, and evaporate when we look at them in more detail. However, this analogy turns out to be deep and strong. Indeed, mathematicians often think of Fourier coefficients as being something like the components of vectors, and use language that further cements this kinship. More details of this fascinating viewpoint are given in the Appendix.

The most time-consuming task in calculating Fourier series is usually the evaluation of the integrals. In practice, scientists often use tables of integrals or computer algebra programs, and in this text we will give some standard integrals that can be used as shortcuts. The two results given below are often useful.

Two useful integrals

$$\int t\sin(at)\,dt = \frac{1}{a^2}\big(\sin(at) - at\cos(at)\big), \qquad (32)$$

$$\int t\cos(at)\,dt = \frac{1}{a^2}\big(\cos(at) + at\sin(at)\big). \qquad (33)$$

These results can be derived using integration by parts.

Example 7

Find the Fourier series for the sawtooth function

$$f(t) = \begin{cases} -t & \text{for } -1 \le t \le 0, \\ t & \text{for } 0 < t < 1, \end{cases}$$

$$f(t+2) = f(t),$$

sketched in Figure 21.

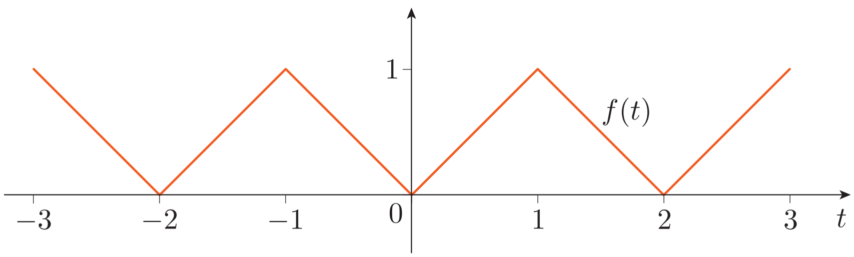


Figure 21 A sawtooth function

Solution

The function f is even and has fundamental period $\tau = 2$. Using equation (17), its Fourier series has the form

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos(n\pi t) + \sum_{n=1}^{\infty} B_n\sin(n\pi t),$$

where A_0, A_n and B_n (for $n = 1, 2, 3, \ldots$) are the Fourier coefficients.

From equation (18) we get

$$A_0 = \frac{1}{2}\int_{-1}^{1} f(t)\,dt.$$

Since f is even, we can use equation (9) to write

$$A_0 = \int_0^1 f(t)\,dt.$$

However, $f(t) = t$ for $0 < t < 1$, hence

$$A_0 = \int_0^1 t\,dt = \left[\tfrac{1}{2}t^2\right]_0^1 = \tfrac{1}{2}.$$

We use a similar method to calculate the A_n:

$$\begin{aligned} A_n &= \frac{2}{2}\int_{-1}^{1} f(t)\cos(n\pi t)\,dt \\ &= 2\int_{0}^{1} f(t)\cos(n\pi t)\,dt \\ &= 2\int_{0}^{1} t\cos(n\pi t)\,dt. \end{aligned}$$

Using the standard integral in equation (33), we get

$$\begin{aligned} A_n &= \frac{2}{(n\pi)^2}\Big[\cos(n\pi t) + n\pi t\sin(n\pi t)\Big]_0^1 \\ &= \frac{2}{(n\pi)^2}\big(\cos(n\pi) + n\pi\sin(n\pi) - 1\big). \end{aligned}$$

These values are given in equations (21) and (22).

But $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$ for any integer n. So

$$A_n = \frac{2}{(n\pi)^2}((-1)^n - 1) \quad (n = 1, 2, 3, \ldots).$$

The B_n are given by

$$B_n = \int_{-1}^{1} f(t)\sin(n\pi t)\,dt.$$

However, $\sin(n\pi t)$ is an odd function of t, and $f(t)$ is an even function of t, so the integrand $f(t)\sin(n\pi t)$ is an odd function of t. From Subsection 1.4 we know that the integral of an odd function over a range that is symmetric about the origin vanishes, so

$$B_n = 0 \quad (n = 1, 2, 3, \ldots).$$

Hence the Fourier series for $f(t)$ is given by

$$F(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}((-1)^n - 1)\cos(n\pi t). \tag{34}$$

The first few terms in equation (34) look like

$$F(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi t) - \frac{4}{9\pi^2}\cos(3\pi t) - \frac{4}{25\pi^2}\cos(5\pi t) - \cdots.$$

To investigate how rapidly the right-hand side of this equation approaches the original function $f(t)$ as we add more and more terms, we introduce the truncated Fourier series $F_N(t)$ defined by

$$F_N(t) = \frac{1}{2} + \sum_{n=1}^{N} \frac{2}{(n\pi)^2}((-1)^n - 1)\cos(n\pi t).$$

Figure 22 compares F_3 (in blue) and F_5 (in red) with the original function f. We see that F_5 is already a very good approximation to f.

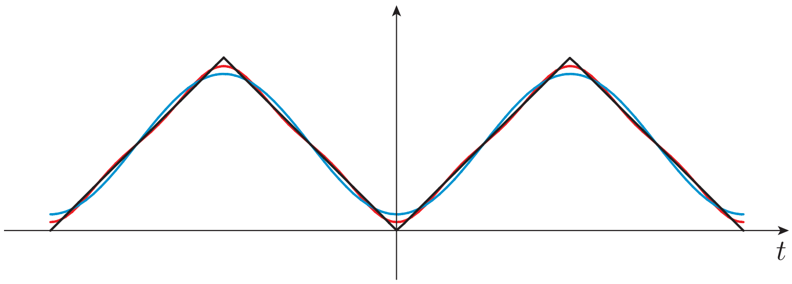


Figure 22 The sawtooth function f (in black) together with its truncated Fourier series approximations F_3 (in blue) and F_5 (in red)

It is also possible to express the Fourier series in a slightly different way. Noting that the non-zero terms in equation (34) occur only for odd values of n (i.e. for the values $n = 2m - 1$ with $m = 1, 2, 3, \ldots$), we can write the Fourier series as

$$F(t) = \frac{1}{2} - \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2\pi^2} \cos((2m-1)\pi t), \tag{35}$$

and this is entirely equivalent to equation (34).

Example 8

Find the Fourier series for the function

$$f(t) = \begin{cases} 0 & \text{for } -1 < t < 0, \\ t & \text{for } 0 \le t \le 1, \end{cases}$$

$$f(t+2) = f(t),$$

sketched in Figure 23.

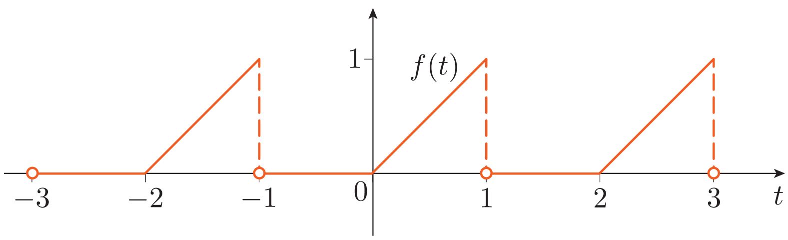


Figure 23 Graph of a function $f(t)$ that is neither even nor odd

Solution

The function $f(t)$ is neither even nor odd, and is discontinuous. It has fundamental period $\tau = 2$. So, using equation (17), its Fourier series takes the form

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi t) + \sum_{n=1}^{\infty} B_n \sin(n\pi t),$$

where the coefficients A_0, A_n and B_n ($n = 1, 2, 3, \ldots$) are to be determined.

From equation (18) we get

$$A_0 = \frac{1}{2}\int_{-1}^{1} f(t)\,dt.$$

To integrate a function defined on pieces, we simply integrate each piece in turn using the correct value of the function on each piece. So

$$A_0 = \frac{1}{2}\int_{-1}^{0} f(t)\,dt + \frac{1}{2}\int_{0}^{1} f(t)\,dt = \frac{1}{2}\int_{0}^{1} t\,dt = \left[\tfrac{1}{4}t^2\right]_0^1 = \tfrac{1}{4},$$

where we have used the fact that $f(t) = 0$ for $-1 < t < 0$ and $f(t) = t$ for $0 \le t \le 1$.

Equation (19) gives

$$A_n = \frac{2}{2}\int_{-1}^{1} f(t)\cos(n\pi t)\,dt \quad (n = 1, 2, 3, \ldots).$$

Again, to integrate a function defined on pieces, we simply integrate on each piece in turn using the correct value of the function on each piece. This gives

$$\begin{aligned} A_n &= \int_{-1}^{0} f(t)\cos(n\pi t)\,dt + \int_{0}^{1} f(t)\cos(n\pi t)\,dt \\ &= \int_{0}^{1} t\cos(n\pi t)\,dt. \end{aligned}$$

Using the standard integral given in equation (33), we get

$$\begin{aligned} A_n &= \frac{1}{(n\pi)^2}\big[\cos(n\pi t) + n\pi t\sin(n\pi t)\big]_0^1 \\ &= \frac{1}{(n\pi)^2}\big(\cos(n\pi) + n\pi\sin(n\pi) - 1\big). \end{aligned}$$

These values are given in equations (21) and (22).

But $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$ for any integer n. So

$$A_n = \frac{1}{(n\pi)^2}((-1)^n - 1) \quad (n = 1, 2, 3, \ldots).$$

The B_n are found using equation (20):

$$\begin{aligned} B_n &= \int_{-1}^{1} f(t)\sin(n\pi t)\,dt \\ &= \int_{-1}^{0} f(t)\sin(n\pi t)\,dt + \int_{0}^{1} f(t)\sin(n\pi t)\,dt \\ &= \int_{0}^{1} t\sin(n\pi t)\,dt. \end{aligned}$$

Now use the standard integral in equation (32). This gives

$$\begin{aligned} B_n &= \frac{1}{(n\pi)^2}\big[\sin(n\pi t) - n\pi t\cos(n\pi t)\big]_0^1 \\ &= \frac{1}{(n\pi)^2}\big(\sin(n\pi) - n\pi\cos(n\pi)\big). \end{aligned}$$

Since $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ for any integer n, we get

$$B_n = -\frac{(-1)^n}{n\pi} \quad (n = 1, 2, 3, \ldots).$$

Putting all these results together, the required Fourier series is

$$F(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}((-1)^n - 1)\cos(n\pi t) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi}\sin(n\pi t).$$

To see how the right-hand side of this equation for $F(t)$ approaches the original function $f(t)$ as we add more and more terms, we once again consider a truncated Fourier series:

$$F_N(t) = \frac{1}{4} + \sum_{n=1}^{N} \frac{1}{(n\pi)^2}((-1)^n - 1)\cos(n\pi t) - \sum_{n=1}^{N} \frac{(-1)^n}{n\pi}\sin(n\pi t).$$

Other definitions could be given for F_N, but this makes no difference to our general argument.

In Figure 24 we compare F_5 (in blue) and F_{20} (in red) with the original function f (in black).

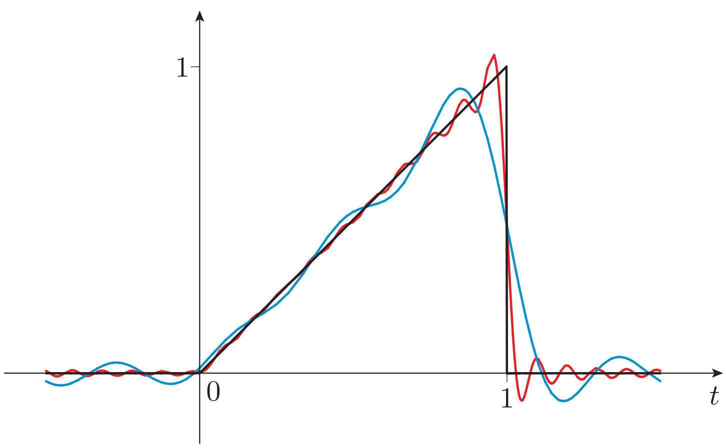


Figure 24 The function f (in black) together with the truncated Fourier series approximations F_5 (in blue) and F_{20} (in red)

Notice that in this case, we need to take $N = 20$ or more to get a reasonable approximation to this function at most points in its domain. A similar observation was made for the square-wave function in Example 6. However, the Fourier series for the continuous function discussed in Example 7 showed quite different behaviour; in that case a truncated Fourier series F_3 with only four terms is already a good approximation to the original function. In general, fewer terms of a Fourier series are needed to approximate a continuous function than a discontinuous one. We say that the Fourier series for a continuous function *converges* more rapidly than the Fourier series for a discontinuous function. We will return to this point in Subsection 3.1.

Fourier series and musical sounds

You can actually hear the difference between different Fourier series! A musical note corresponds to a rapid oscillation in the pressure of air, with a given fundamental period τ, and a corresponding fundamental angular frequency $\omega = 2\pi/\tau$. For example, middle C has a fundamental period of about 3.82 milliseconds. All musical instruments playing this note produce pressure variations that are periodic functions of time with this fundamental period. But the precise shapes of these periodic functions are *different* when produced by a piano, a guitar or a violin, and the corresponding Fourier series are different too.

The Fourier series for a musical note is a sum of sinusoidal functions with frequencies that are *harmonics*, that is, integer multiples of the fundamental angular frequency of the musical tone. The greater the contribution from the harmonics, the 'brighter' the tone of the musical instrument. For example, a violin produces a brighter tone than a guitar because the oscillation has a greater contribution from high harmonics of the fundamental frequency. The periodic function in Figure 24 contains harmonics that die away slowly as n increases. If this function represented a musical note of middle C, the effect would be unpleasantly rasping.

Figure 25 The first commercial Moog synthesiser (1964)

What can be analysed can also be synthesised. Figure 25 shows a Moog synthesiser, which was used in 1960s and 1970s pop music to construct complex musical tones from linear combinations of sinusoidal oscillations (Fourier series, in fact!).

Exercise 11

Although it can be tedious, aim to be patient and careful when evaluating the integrals needed in Fourier series!

Find the Fourier series for the function

$$f(t) = t \quad \text{for } -\pi \le t < \pi,$$
$$f(t + 2\pi) = f(t).$$

Exercise 12

Find the Fourier series for the function

$$f(x) = \begin{cases} x + 1 & \text{for } -1 \le x \le 0, \\ 1 & \text{for } 0 < x < 1, \end{cases}$$
$$f(x + 2) = f(x).$$

2.2 Fourier series for odd and even functions

Examples 6–8 in the previous subsection displayed three different types of behaviour. In Example 6, the function was odd and the Fourier series contained only sine terms. In Example 7, the function was even and the Fourier series contained only constant and cosine terms. Finally, in Example 8, the function was neither odd nor even and its Fourier series contained constant, cosine and sine terms.

This pattern is easily explained. If the function $f(t)$ is odd, the integrals for A_0 and A_n in equations (18) and (19) involve odd integrands integrated over a range that is symmetric about the origin: such integrals are equal to zero, leaving only sine terms in the Fourier series. Similarly, if the function $f(t)$ is even, the integrals for B_n in equation (20) involve odd integrands integrated over a range that is symmetric about the origin: these integrals vanish, leaving only constant and cosine terms in the Fourier series. Taking note of these facts, the procedure for calculating the Fourier series for a function $f(t)$ can be simplified if $f(t)$ is either odd or even.

If $f(t)$ is an odd periodic function, with fundamental period τ, then all the Fourier coefficients A_0 and A_n vanish, and using equations (9) and (20), the integral for the B_n coefficients can be simplified slightly:

$$B_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt = \frac{4}{\tau}\int_0^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt.$$

The second integral is often a little easier to evaluate than the first, but both lead to the same result.

Procedure 2 Fourier series for odd periodic functions

To find the Fourier series for an odd periodic function $f(t)$, proceed as follows.

1. Identify $f(t)$ as being odd, and find its fundamental period τ.
2. Write down the Fourier series
$$F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right). \tag{36}$$
3. Find the coefficients by evaluating the definite integrals
$$B_n = \frac{4}{\tau}\int_0^{\tau/2} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots). \tag{37}$$
4. If desired, express the final Fourier series in a compact form with general formulas for its coefficients.

A similar simplification occurs for even functions. In this case, the B_n coefficients vanish. Also, the integrands for A_0 and A_n are even, so these coefficients can be expressed as integrals from 0 to $\tau/2$.

Procedure 3 Fourier series for even periodic functions

To find the Fourier series for an even periodic function $f(t)$, proceed as follows.

1. Identify $f(t)$ as being even, and find its fundamental period τ.
2. Write down the Fourier series
$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right). \tag{38}$$
3. Find the coefficients by evaluating the definite integrals
$$A_0 = \frac{2}{\tau}\int_0^{\tau/2} f(t)\,dt, \tag{39}$$
$$A_n = \frac{4}{\tau}\int_0^{\tau/2} f(t)\cos\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots). \tag{40}$$
4. If desired, express the final Fourier series in a compact form with general formulas for its coefficients.

To take advantage of the oddness and evenness of functions, we must take the fundamental interval to be symmetric about the origin, from $-\tau/2$ to $\tau/2$. However, a piecewise continuous periodic function may be given on some other fundamental interval. For example, the function $h(t)$ in Exercise 14 below is defined on $-\frac{1}{2}\pi \le t < \frac{3}{2}\pi$, with a periodicity condition that gives the values of the function elsewhere. In cases like this, it is advisable to sketch a graph of the function – partly to check that it is odd or even, and partly to get the values that are needed in the range $-\tau/2 \le t < \tau/2$.

Example 9

Use Procedure 2 to find the Fourier series for the sawtooth function
$$p(t) = \begin{cases} t & \text{for } -\frac{1}{2} \le t \le \frac{1}{2}, \\ 1 - t & \text{for } \frac{1}{2} < t < \frac{3}{2}, \end{cases}$$
$$p(t+2) = p(t).$$

This function was sketched in Figure 3, which is reproduced in Figure 26.

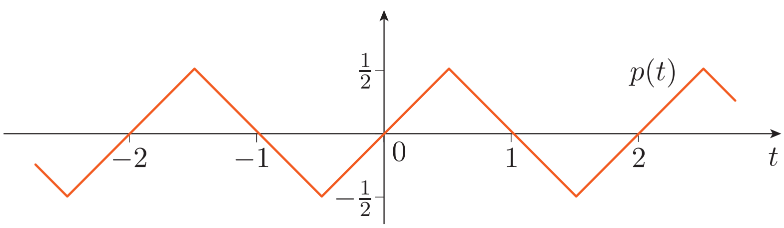


Figure 26 A sawtooth function

Solution

It is clear from Figure 26 that $p(t)$ is an odd function with fundamental period $\tau = 2$. Following Procedure 2, its Fourier series contains only sine terms and takes the form

$$P(t) = \sum_{n=1}^{\infty} B_n \sin(n\pi t).$$

The Fourier coefficients are given by

$$B_n = \frac{4}{2}\int_0^1 p(t)\sin(n\pi t)\,dt.$$

Within this range of integration, $p(t) = t$ for $0 \le t \le 1/2$, and $p(t) = 1 - t$ for $1/2 < t \le 1$. So the integral splits into two pieces:

$$B_n = 2\int_0^{1/2} t\sin(n\pi t)\,dt + 2\int_{1/2}^1 (1-t)\sin(n\pi t)\,dt.$$

Using the standard integral in equation (32), we obtain

$$\begin{aligned} B_n = {} & \frac{2}{(n\pi)^2}\Big[\sin(n\pi t) - n\pi t\cos(n\pi t)\Big]_0^{1/2} + \frac{2}{n\pi}\Big[-\cos(n\pi t)\Big]_{1/2}^1 \\ & - \frac{2}{(n\pi)^2}\Big[\sin(n\pi t) - n\pi t\cos(n\pi t)\Big]_{1/2}^1. \end{aligned}$$

Substituting in the limits, we get

$$\begin{aligned} B_n = {} & \frac{2}{(n\pi)^2}\left(\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right) \\ & + \frac{2}{n\pi}\left(-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right)\right) \\ & - \frac{2}{(n\pi)^2}\left(\sin(n\pi) - n\pi\cos(n\pi) - \sin\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right). \end{aligned}$$

Carefully combining terms, making cancellations, and using $\sin(n\pi) = 0$, this gives

$$B_n = \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right).$$

The required Fourier series is therefore

$$P(t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin(n\pi t).$$

For $n = 1, 2, 3, 4, 5, 6, 7$, the values of $\sin(n\pi/2)$ are $1, 0, -1, 0, 1, 0, -1$, so the first few terms in the Fourier series are

$$P(t) = \frac{4}{\pi^2}\left(\sin(\pi t) - \tfrac{1}{9}\sin(3\pi t) + \tfrac{1}{25}\sin(5\pi t) - \tfrac{1}{49}\sin(7\pi t) + \cdots\right),$$

as stated in equation (2) of the Introduction.

Only the odd values of n contribute to the Fourier series. Putting $n = 2m - 1$ and noting that equation (24) gives $\sin((2m-1)\pi/2) = (-1)^{m+1}$, this Fourier series can also be written in the alternative form

$$P(t) = \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin((2m-1)\pi t).$$

Exercise 13

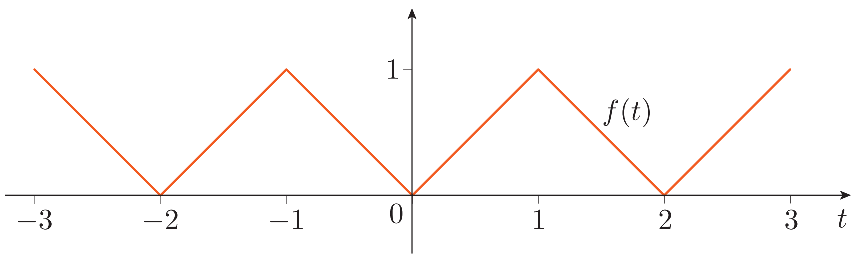
This function was discussed in Example 7, but the present calculation is more efficient.

Use Procedure 3 to find the Fourier series for the sawtooth function

$$f(t) = \begin{cases} -t & \text{for } -1 \le t \le 0, \\ t & \text{for } 0 < t < 1, \end{cases}$$

$$f(t+2) = f(t),$$

sketched in the figure below.



Exercise 14

Find the Fourier series for the square-wave function defined in equation (5) by

$$h(t) = \begin{cases} 1 & \text{for } -\frac{1}{2}\pi \le t \le \frac{1}{2}\pi, \\ 0 & \text{for } \frac{1}{2}\pi < t < \frac{3}{2}\pi, \end{cases}$$

$$h(t+2\pi) = h(t).$$

Exercise 15

Find the Fourier series for the function $c(t)$ defined in the Introduction as

$$c(t) = |\cos t|,$$

and sketched in Figure 2.

You may use the standard integral

$$\int \cos(at)\cos(bt)\,dt = \frac{b\cos(at)\sin(bt) - a\cos(bt)\sin(at)}{b^2 - a^2} \quad (a \ne b).$$

2.3 Functions defined over a finite domain

So far you have seen how to calculate the Fourier series for any *periodic* function. However, this is not the whole story. It is also possible to calculate the Fourier series for (almost) *any* function, provided that it is defined over a finite domain. This idea will be particularly useful in the next unit.

Suppose that a function $f(t)$ is defined within the finite interval $0 \leq t \leq T$ of length T (see Figure 27). Furthermore, suppose that we do not care about what happens to the function outside this interval.

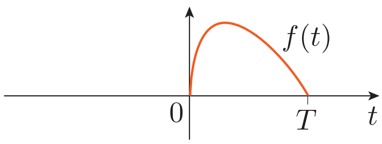


Figure 27 A function $f(t)$ defined on a finite interval $0 \leq t \leq T$

Then we can always define another function $f_{\text{ext}}(t)$ to be equal to $f(t)$ on the interval $0 \leq t \leq T$, and to be periodic with fundamental period T everywhere else. This function is called a **periodic extension** of $f(t)$ and is written as

$$f_{\text{ext}}(t) = f(t) \quad \text{for } 0 \leq t < T,$$
$$f_{\text{ext}}(t+T) = f_{\text{ext}}(t).$$

The graph of $f_{\text{ext}}(t)$ consists of copies of $f(t)$ shifted by T and by all positive and negative integer multiples of T, and is shown in Figure 28.

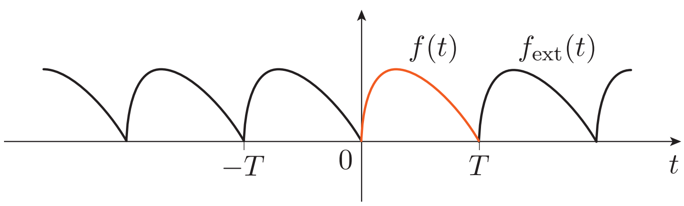


Figure 28 A periodic extension $f_{\text{ext}}(t)$ of $f(t)$

The periodic extension $f_{\text{ext}}(t)$ is a periodic function of fundamental period T, and we can find its Fourier series as normal. The resulting Fourier series will be equal to $f_{\text{ext}}(t)$ everywhere, and is equal to $f(t)$ for $0 \leq t \leq T$. So this Fourier series represents the non-periodic function $f(t)$ inside its domain of definition, $0 \leq t \leq T$. The periodic extension shown in Figure 28 is neither even nor odd, so the Fourier series contains both sine and cosine terms.

With a little preparation, we can use $f(t)$ to construct periodic functions that are either even or odd, before extending over all t. This is generally a sensible thing to do because the resulting Fourier series will be simpler. We make the following definitions.

Even and odd periodic extensions

Consider a function $f(t)$ defined over a finite domain $0 \le t \le T$.

The **even periodic extension** of $f(t)$ is given by

$$f_{\text{even}}(t) = \begin{cases} f(t) & \text{for } 0 \le t \le T, \\ f(-t) & \text{for } -T < t < 0, \end{cases}$$

$$f_{\text{even}}(t + 2T) = f_{\text{even}}(t).$$

An example of this extension is shown in Figure 29.

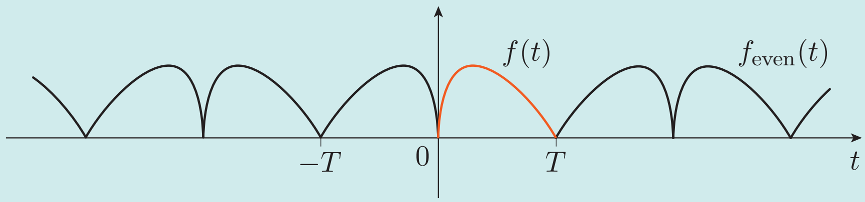


Figure 29 The even periodic extension $f_{\text{even}}(t)$ of $f(t)$

The **odd periodic extension** of $f(t)$ is given by

$$f_{\text{odd}}(t) = \begin{cases} f(t) & \text{for } 0 \le t \le T, \\ -f(-t) & \text{for } -T < t < 0, \end{cases}$$

$$f_{\text{odd}}(t + 2T) = f_{\text{odd}}(t).$$

An example of this extension is shown in Figure 30.

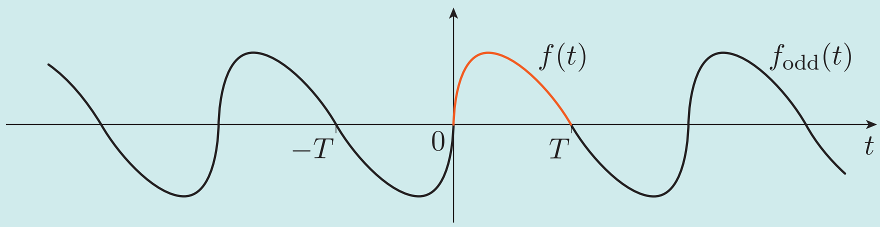


Figure 30 The odd periodic extension $f_{\text{odd}}(t)$ of $f(t)$

In general, both these functions have fundamental period $\tau = 2T$.

In exceptional cases, the even periodic extension may have fundamental period $\tau = T$ (see Exercise 17).

Example 10

Find the even and odd periodic extensions of the function

$$f(t) = t \quad \text{for } 0 \le t \le 1,$$

and sketch these two extensions.

Solution

The even periodic extension is given by

$$f_{\text{even}}(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1, \\ -t & \text{for } -1 < t < 0, \end{cases}$$

$$f_{\text{even}}(t+2) = f_{\text{even}}(t).$$

This function is sketched in Figure 31, with the original function shown in orange.

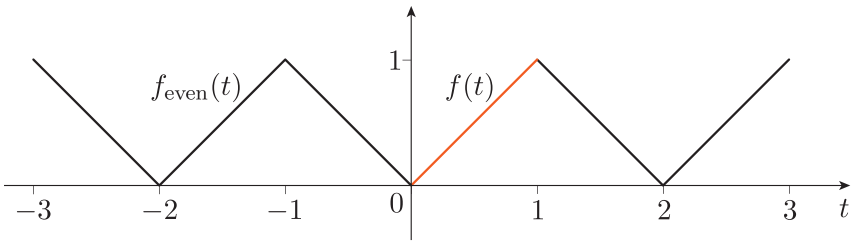


Figure 31 Even periodic extension

The odd periodic extension is given by

$$f_{\text{odd}}(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1, \\ t & \text{for } -1 < t < 0, \end{cases}$$

$$f_{\text{odd}}(t+2) = f_{\text{odd}}(t).$$

This function is sketched in Figure 32, with the original function shown in orange.

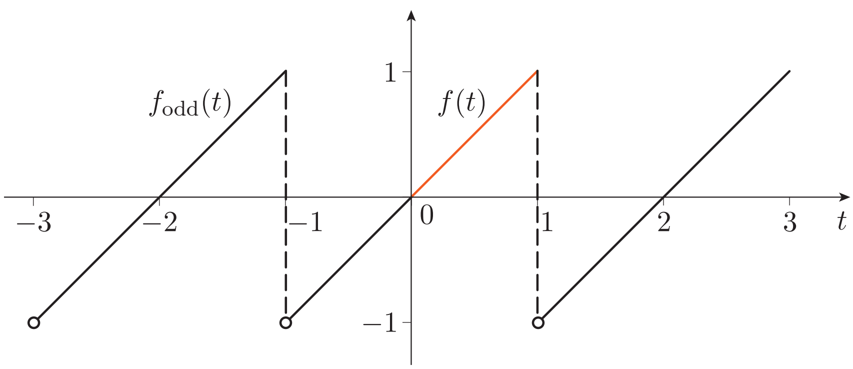


Figure 32 Odd periodic extension

In this particular case, both extensions can be expressed in alternative forms. The even periodic extension is

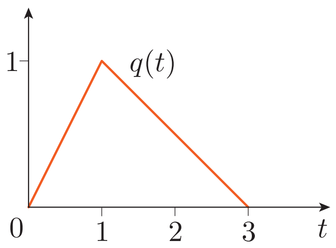
$$f_{\text{even}}(t) = |t| \quad \text{for } -1 < t \leq 1,$$

$$f_{\text{even}}(t+2) = f_{\text{even}}(t),$$

and the odd periodic extension is

$$f_{\text{odd}}(t) = t \quad \text{for } -1 < t \leq 1,$$

$$f_{\text{odd}}(t+2) = f_{\text{odd}}(t).$$



Exercise 16

Consider the function

$$q(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le 3, \end{cases}$$

sketched in the margin.

Define the even and odd periodic extensions of $q(t)$, simplifying the formulas if possible. State the fundamental periods, and sketch each extension over a range of three periods.

The following important example illustrates how a function defined on a finite interval can be represented by a Fourier series.

Example 11

The function

$$f(x) = \begin{cases} \dfrac{2d}{L}x & \text{for } 0 \le x \le L/2, \\ \dfrac{2d}{L}(L - x) & \text{for } L/2 < x \le L, \end{cases}$$

where d and L are positive constants, is defined on the finite interval $0 \le x \le L$. Express $f(x)$ as a Fourier series that involves only sine terms.

(*Hint*: With a change of variable to $u = x/L$, the integrals needed for the Fourier coefficients can be related to those calculated in Example 9.)

Solution

Because we are looking for a Fourier series that involves only sine terms, we need to consider the *odd* periodic extension of $f(x)$, denoted by $f_{\text{odd}}(x)$. This is sketched in Figure 33.

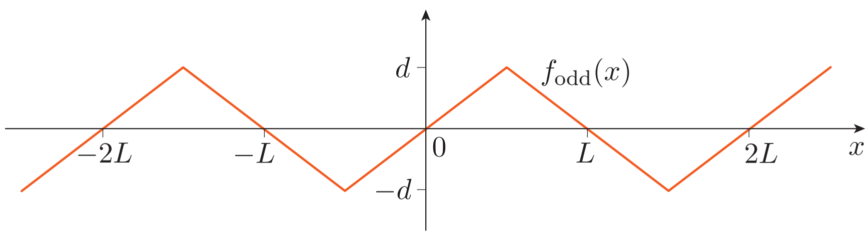


Figure 33 The odd periodic extension of the function $f(x)$

The function $f_{\text{odd}}(x)$ is odd and has period $\tau = 2L$, so its Fourier series takes the form

$$F_{\text{odd}}(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

where the Fourier coefficients B_n are given by

$$B_n = \frac{4}{2L}\int_0^L f_{\text{odd}}(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

But $f_{\text{odd}}(x) = f(x)$ on the interval $0 \le x \le L$, so

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Using the piecewise definition of $f(x)$ given in the question, we obtain

$$B_n = \frac{4d}{L^2}\left[\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right].$$

Making the suggested substitution $u = x/L$, we get

$$B_n = 4d\left[\int_0^{1/2} u\sin(n\pi u)\,du + \int_{1/2}^{1}(1-u)\sin(n\pi u)\,du\right].$$

The required integrals have already been evaluated in Example 9. Comparing with the working in that example, we conclude that

$$B_n = \frac{8d}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \quad (n = 1, 2, 3, \ldots).$$

So

$$F_{\text{odd}}(x) = \frac{8d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Since $f(x)$ and $f_{\text{odd}}(x)$ coincide on the interval $0 \leq x \leq L$, this is the required sine Fourier series $F(x)$ for $f(x)$.

For $n = 1, 2, 3, 4, 5, 6, 7$, the values of $\sin(n\pi/2)$ are $1, 0, -1, 0, 1, 0, -1$, so the first few terms in the Fourier series are

$$F(x) = \frac{8d}{\pi^2}\left(\sin\left(\frac{\pi x}{L}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{L}\right) - \frac{1}{7^2}\sin\left(\frac{7\pi x}{L}\right) + \cdots\right).$$

Exercise 17

Consider the same function $f(x)$ as that discussed in Example 11. Within its domain of definition, $0 \leq x \leq L$, represent this function by a Fourier series that involves only constant and cosine functions.

To represent the original function $f(x)$ in Example 11 by a Fourier series, we can use the odd periodic extension, obtaining a series that contains only sine terms (as in Example 11), or we can use the even periodic extension, obtaining a series that contains only constant and cosine terms (as in Exercise 17).

Sometimes one choice is better than the other. In general, if we want to approximate a function by a truncated Fourier series, it is better to use a periodic extension that is continuous, rather than discontinuous. This is because, as pointed out earlier, the Fourier series for a continuous function converges more rapidly than that of a discontinuous function. So for the function discussed in Example 10 we would use the even periodic extension to obtain the Fourier series.

However, in the next unit we will use Fourier series to solve partial differential equations, and in that case our choice of an even or odd periodic extension is generally dictated by other factors, namely the boundary conditions.

Exercise 17 is an exceptional case in which the even periodic extension has fundamental period $\tau = L$ rather than $\tau = 2L$. This simplifies the calculations because we need integrals only over the range from 0 to $L/2$. If we were to treat the function in Exercise 17 as having period $2L$, then the usual formula would eventually give the *same* Fourier series, although the calculations would be longer. In general, making the mistake of using a non-fundamental period rather than the fundamental period will always give the same final Fourier series, but at the expense of more labour.

Exercise 18

Consider the function

$$f(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{\pi}{2}, \\ -1 & \text{for } \frac{\pi}{2} < t \leq \pi. \end{cases}$$

(a) Define the even periodic extension, simplifying your answer as much possible. Sketch this function over $-3\pi/2 \leq t \leq 3\pi/2$, and state its fundamental period.

(b) Find the Fourier series for the even periodic extension.

(c) Define the odd periodic extension. Sketch this function over $-\pi \leq t \leq 3\pi$, and state its fundamental period.

(d) By slightly changing the definition of the odd extension at points of discontinuity, create a periodic extension that has fundamental period π, and hence find the Fourier series. (*Hint*: You may find the result of Example 6 useful.)

3 Working with Fourier series

This section contains a miscellany of topics related to Fourier series. First, it takes a closer look at the fact that the truncated Fourier series for discontinuous functions are slow to converge, especially near the points of discontinuity. It then goes on to look at some useful techniques for calculating and manipulating Fourier series, including differentiating them.

3.1 The Gibbs phenomenon

You saw in Section 2 that a continuous function can be closely approximated by the first few terms of its Fourier series. As more and more terms are added, the truncated Fourier series and the original

function become practically identical in value everywhere. For example, a truncated Fourier series containing only a few terms gives a good approximation for the continuous function in Figure 22.

A different situation applies to discontinuous functions. For a given number of terms in the truncated Fourier series, the approximation is worse, especially around the points of discontinuity. In Figure 34(a) we give a comparison of $F_{20}(t)$ with a square-wave function, while in Figure 34(b) we magnify a portion of this graph near the discontinuous point $t = 0$. Note that F_{20} deviates from the original function by oscillating around it.

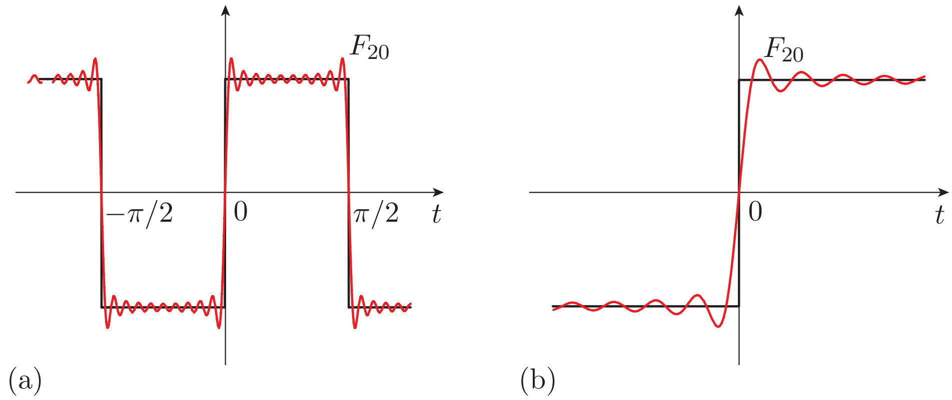


Figure 34 Comparison of a square-wave function (in black) with F_{20} (in red): (a) the first 20 terms in the Fourier series; (b) a magnification in the vicinity of a discontinuity

Such a deviation occurs in the vicinity of *any* discontinuous function and is known as the **Gibbs phenomenon**, after the American mathematical physicist Josiah Gibbs (Figure 35).

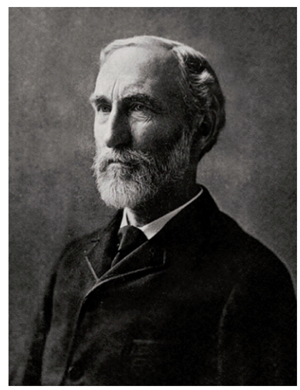


Figure 35 Josiah Gibbs (1839–1903)

The main points of the Gibbs phenomenon are as follows. The truncated Fourier series $F_N(t)$ (with N terms, where N is large) provides a good approximation to $f(t)$ at points well away from any discontinuity. However, in the region of a discontinuity, F_N oscillates around f as shown in Figure 34. As N increases, F_N provides a better approximation to f, and the deviations are pushed into a region closer and closer to the discontinuity, but their amplitude does not diminish. Most importantly, at the point of discontinuity, as N tends to infinity, F_N always takes the average value of the function on either side of the discontinuity. We highlight this crucial point below.

The value of a Fourier series at a point of discontinuity

If $f(t)$ is discontinuous at $t = t_0$, then the value of the corresponding Fourier series $F(t)$ at this point is the average of the function values immediately above and below the discontinuity. That is,

$$F(t_0) = \tfrac{1}{2}\left(f(t_0^+) + f(t_0^-)\right),$$

where $f(t_0^+)$ is the value of $f(t)$ just above t_0, and $f(t_0^-)$ is the value of $f(t)$ just below t_0.

In fact, this rule also works at points where f is continuous. If f is continuous at t_0, then $f(t_0^-) = f(t_0^+) = f(t_0)$ and hence

$$F(t_0) = \tfrac{1}{2}\left(f(t_0^+) + f(t_0^-)\right) = f(t_0),$$

so the Fourier series converges to $f(t_0)$ as expected.

History of the Gibbs phenomenon

This description closely follows C. Lanczos (1966) *Discourse on Fourier Series*, Oliver and Boyd.

The American experimental physicist Albert Michelson (Figure 36) is primarily known for his 1887 experiment with Edward Morley, which showed that the speed of light is independent of the direction in which it is measured. This result undermined the concept of an ether, and prepared the ground for Einstein's special theory of relativity.

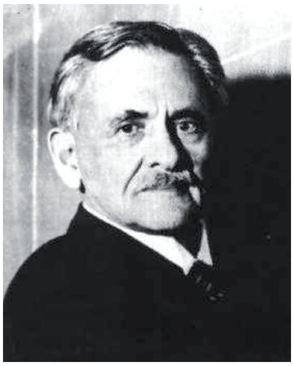


Figure 36 Albert Michelson (1852–1931)

However, Michelson also invented many physical instruments of high precision. In 1898 he constructed a mechanical machine that could compute the first 80 Fourier coefficients for a function that was described numerically; the machine could also plot a graph of the truncated Fourier series and compare this with a graph of the original function. Michelson found that in most cases the input function and the truncated Fourier series agreed well everywhere. But for a discontinuous function, the truncated Fourier series agreed well *except near the point of discontinuity*. Michelson was puzzled and wrote to Gibbs, who explained the phenomenon mathematically and published his findings in Volume 59 of *Nature* (1898–9, pp. 200 and 606).

Although the phenomenon is named after Gibbs, it was first noticed and explained half a century earlier, in 1848, by the obscure English mathematician Henry Wilbraham. This work was unknown to Gibbs.

Example 12

What is the value of the Fourier series for the square-wave function $h(t)$ given in Exercise 14 (depicted here in Figure 37), at the point $t = \pi/2$?

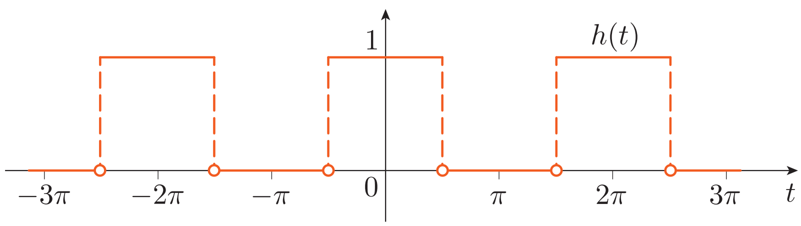


Figure 37 The square-wave function $h(t)$

Solution

The function $h(t)$ is discontinuous at $t = \pi/2$. Just below $t = \pi/2$, $h(t) = 1$, and just above $t = \pi/2$, $h(t) = 0$. Hence at $t = \pi/2$ the Fourier series converges to

$$H(\pi/2) = \tfrac{1}{2}(0 + 1) = \tfrac{1}{2}.$$

We can compare this result with the Fourier series derived in Exercise 14. There we showed that

$$H(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos((2n-1)t).$$

But at $t = \pi/2$, $\cos((2n-1)t) = 0$ for n an integer, so $H(t) = \frac{1}{2}$, in agreement with the average value derived above.

Exercise 19

(a) What is the value of the Fourier series for the function $f(t)$ given in Example 8, at the point $t = 1$?

(b) By comparing with the Fourier series at $t = 1$, show that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

3.2 Shifting the range of integration

Suppose that we have a function $g(t)$ with period τ. Then the integral

$$I = \int_{-\tau/2}^{\tau/2} g(t)\, dt$$

is just the area under the graph of the function on the fundamental interval $-\tau/2 \le t \le \tau/2$, as shown in Figure 38. If we shift the interval to $-\tau/2 + a \le t \le \tau/2 + a$, as in Figure 39, we see that the area lost on the left is equal to the area gained on the right. Therefore

$$I = \int_{-\tau/2}^{\tau/2} g(t)\, dt = \int_{-\tau/2+a}^{\tau/2+a} g(t)\, dt.$$

In other words, when we integrate a function with period τ over an interval of length τ, it doesn't matter which interval of length τ we use.

In our formulas for the Fourier coefficients for a periodic function f with fundamental period τ (equations (18)–(20)), the integrand is always periodic with period τ. Hence those equations are equivalent to the following.

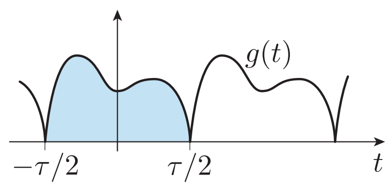


Figure 38 Area under g on the interval $-\tau/2 \le t \le \tau/2$

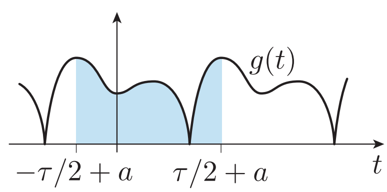


Figure 39 Area under g on the interval $-\tau/2 + a \le t \le \tau/2 + a$

Alternative formulas for Fourier coefficients

The Fourier coefficients for any function f with fundamental period τ are given by

$$A_0 = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} f(t)\,dt, \tag{41}$$

$$A_n = \frac{2}{\tau}\int_{t_0}^{t_0+\tau} f(t)\cos\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots), \tag{42}$$

$$B_n = \frac{2}{\tau}\int_{t_0}^{t_0+\tau} f(t)\sin\left(\frac{2n\pi t}{\tau}\right)dt \quad (n = 1, 2, 3, \ldots), \tag{43}$$

where t_0 can take any value.

Equations (41)–(43) are sometimes easier to use than equations (18)–(20), as the following exercise illustrates.

Exercise 20

Show that the Fourier series for the function

$$f(x) = x \quad \text{for } \tfrac{1}{4} < x \leq \tfrac{3}{4},$$

$$f(x+1) = f(x),$$

is given by

$$F(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left(\sin\left(\tfrac{3}{2}n\pi\right)\cos\left(2n\pi x\right) - \cos\left(\tfrac{3}{2}n\pi\right)\sin\left(2n\pi x\right)\right).$$

3.3 Differentiating Fourier series

Suppose that the continuous function $f(t)$ has the Fourier series

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos(\omega_n t) + \sum_{n=1}^{\infty} B_n\sin(\omega_n t),$$

where A_0, A_n, B_n and ω_n are constants.

Then we can ask: what is the Fourier series for the derivative $f'(t) = df/dt$? The answer is obtained by differentiating each term in $F(t)$ in turn. So $f'(t)$ has the Fourier series

$$F'(t) = -\sum_{n=1}^{\infty} \omega_n A_n\sin(\omega_n t) + \sum_{n=1}^{\infty} \omega_n B_n\cos(\omega_n t).$$

The results in this subsection rely on the function being continuous.

Differentiation of Fourier series

If a *continuous* periodic function $f(t)$ with fundamental period τ has the Fourier series $F(t)$, then its derivative $f'(t)$ has the same fundamental period τ, and its Fourier series is given by $F'(t)$.

This allows us to deduce the Fourier series for $f'(t)$ from the Fourier series for $f(t)$. This may not always be useful because the Fourier series for $f'(t)$ is often easier to find than that for $f(t)$. However, we sometimes need both these Fourier series; in such a case, it is possible to save some time by calculating the Fourier series for $f(t)$ first, and then differentiating it.

Example 13

In equation (35) we showed that the function

$$f(t) = \begin{cases} -t & \text{for } -1 \le t \le 0, \\ t & \text{for } 0 < t < 1, \end{cases}$$

$$f(t+2) = f(t),$$

has Fourier series

$$F(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2} \cos((2n-1)\pi t).$$

This function is shown in Figure 40.

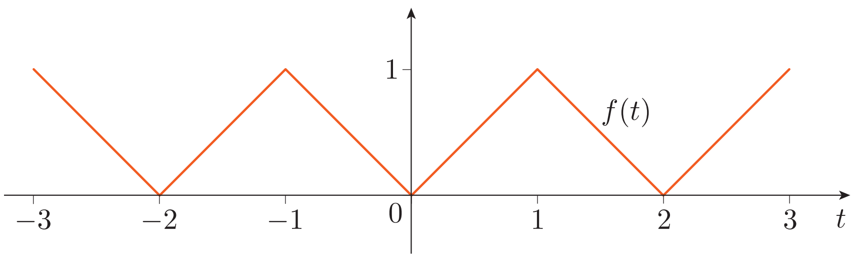


Figure 40 The function $f(t)$

Deduce the Fourier series for the function $f'(t)$, given by

$$f'(t) = \begin{cases} -1 & \text{for } -1 < t < 0, \\ 1 & \text{for } 0 < t < 1, \end{cases}$$

$$f'(t+2) = f'(t),$$

and sketched in Figure 41.

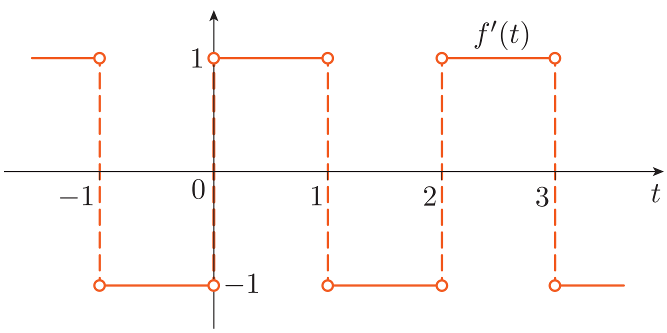


Figure 41 The function $f'(t)$

Solution

The function $f(t)$ is continuous, as can be seen from its graph, and $f'(t)$ is its derivative. The Fourier series for $f'(t)$ is given by the derivative of $F(t)$:

$$\begin{aligned} F'(t) &= 0 - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2} \frac{d}{dt}\big(\cos((2n-1)\pi t)\big) \\ &= \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin((2n-1)\pi t). \end{aligned}$$

In this case, $f'(t)$ is undefined at the points $t = 0, \pm 1, \pm 2, \ldots$ because the slope of the function $f(t)$ changes abruptly there. This does not affect the evaluation of the Fourier series $F'(t)$, which converges to the average value of $f'(t)$ either side of these points (e.g. $F'(1) = F'(2) = \ldots = 0$).

Exercise 21

The function $c(t) = |\cos t|$ can be defined as

$$\begin{aligned} &c(t) = \cos t \quad \text{for } -\tfrac{\pi}{2} < t < \tfrac{\pi}{2}, \\ &c(t+\pi) = c(t). \end{aligned}$$

From Exercise 15, this function has the Fourier series

$$C(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} \cos(2nt).$$

Deduce the Fourier series for the discontinuous function

$$\begin{aligned} &s(t) = \sin t \quad \text{for } -\tfrac{\pi}{2} < t < \tfrac{\pi}{2}, \\ &s(t+\pi) = s(t). \end{aligned}$$

If you are very short of time, you may omit the rest of this unit.

4 The exponential Fourier series

This section will not be assessed in the exam or in continuous assessment. However, you are advised to study it because it contains ideas that are very useful in the physical sciences.

4.1 The exponential Fourier series

There is another form of the Fourier series that is used extensively in mathematics and the physical sciences: this is the *exponential Fourier series*. It is fully equivalent to the *trigonometric Fourier series* discussed so far, but uses the (complex) exponential function instead of sines and

cosines. This series has the advantage of being simpler and more concise than the trigonometric Fourier series, and it is often easier to evaluate its Fourier coefficients. Also, the exponential Fourier series is the starting point for *Fourier transforms*, an advanced topic that is used extensively in applied mathematics and science.

Recall that a function $f(t)$ with fundamental period τ has a trigonometric Fourier series $F(t)$ of the form

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega_n t) + \sum_{n=1}^{\infty} B_n \sin(\omega_n t), \tag{44}$$

where A_0, A_n and B_n are constants, and $\omega_n = 2n\pi/\tau$ for $n = 1, 2, 3, \ldots$.

You know that Euler's formula relates the complex exponential function to cosines and sines, telling us that

$$e^{i\omega_n t} = \cos(\omega_n t) + i\sin(\omega_n t),$$

and conversely,

$$\cos(\omega_n t) = \frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2}, \quad \sin(\omega_n t) = \frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i}.$$

Substituting these results into equation (44) gives

$$F(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2}\left(e^{i\omega_n t} + e^{-i\omega_n t}\right) + \sum_{n=1}^{\infty} \frac{B_n}{2i}\left(e^{i\omega_n t} - e^{-i\omega_n t}\right),$$

and we can then collect terms to get

$$F(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n - iB_n}{2}\, e^{i\omega_n t} + \sum_{n=1}^{\infty} \frac{A_n + iB_n}{2}\, e^{-i\omega_n t}.$$

For $n = 0, 1, 2, 3, \ldots$, we then define the constants

$$C_0 = A_0, \quad C_n = \frac{A_n - iB_n}{2}, \quad C_{-n} = \frac{A_n + iB_n}{2}. \tag{45}$$

Notice that in the last of these definitions, the index used to label the constant C_{-n} is negative. For example, $C_{-1} = (A_1 + iB_1)/2$. Using this notation, the Fourier series can be written as

$$F(t) = C_0 + \sum_{n=1}^{\infty} C_n\, e^{i\omega_n t} + \sum_{n=1}^{\infty} C_{-n}\, e^{-i\omega_n t}.$$

Recalling that $\omega_n = 2n\pi/\tau$, this can also be expressed as

$$F(t) = C_0 + \sum_{n=1}^{\infty} C_n\, e^{2in\pi t/\tau} + \sum_{n=1}^{\infty} C_{-n}\, e^{-2in\pi t/\tau}. \tag{46}$$

The final sum on the right-hand side is a sum of terms

$$C_{-1}\, e^{-2i\pi t/\tau} + C_{-2}\, e^{-4i\pi t/\tau} + C_{-3}\, e^{-6i\pi t/\tau} + \cdots,$$

and this can be written as a sum over *negative* integers:

$$\sum_{n=-1}^{-\infty} C_n\, e^{2in\pi t/\tau}.$$

Hence we can combine all the terms in equation (46) to get

$$F(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{2in\pi t/\tau}, \tag{47}$$

where the sum is now over all the integers – positive, zero and negative. This is called the **exponential Fourier series**, and the constants C_n are called *exponential Fourier coefficients*.

Equation (47) is more compact than equation (44), but a price has been paid. The sum now extends over all the integers, rather than just the positive integers, and the coefficients C_n are, in general, complex numbers (assuming that A_n and $B_n \neq 0$ are real). Nevertheless, the exponential Fourier series has some clear advantages. Given a function $f(t)$ with fundamental period τ, it turns out that the Fourier coefficients C_n are given by the formula

This result is proved in the Appendix.

$$C_n=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, e^{-2in\pi t/\tau}\, dt \quad (n=0,\pm 1,\pm 2,\ldots).$$

This single formula compares favourably with the three separate integrals needed for A_0, A_n and B_n.

The results obtained above are summarised in the following box.

Exponential Fourier series for periodic functions

For a periodic function $f(t)$, with fundamental period τ, the exponential Fourier series is

$$F(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{2in\pi t/\tau}. \tag{48}$$

The Fourier coefficients C_n are calculated from

$$C_n=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, e^{-2in\pi t/\tau}\, dt \quad (n=0,\pm 1,\pm 2,\ldots). \tag{49}$$

Even if we ultimately want to find a *trigonometric* Fourier series, it can make sense to begin with the exponential Fourier series, using equation (49) to find the C_n, and then use the inverse of equations (45), namely

$$A_0=C_0, \quad A_n=C_n+C_{-n}, \quad B_n=i(C_n-C_{-n}) \quad (n\geq 1), \tag{50}$$

to find A_0, A_n and B_n.

$\overline{f}$ denotes the complex conjugate of f.

If $f(t)$ is real, then we have $\overline{f}(t)=f(t)$ and equation (49) gives

$$\overline{C_n}=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, e^{+2in\pi t/\tau}\, dt = C_{-n}.$$

Combining this with equations (50), we obtain the following.

Fourier coefficients for real functions

$$A_0 = C_0, \quad A_n = 2\,\mathrm{Re}(C_n), \quad B_n = -2\,\mathrm{Im}(C_n) \quad (n \geq 1). \tag{51}$$

Example 14

(a) Find the exponential Fourier series for the function

$$f(t) = t \quad \text{for } -\pi \leq t < \pi,$$
$$f(t+2\pi) = f(t).$$

Hint: You may find the following standard integral useful:

$$\int x\,e^{ax}\,dx = \frac{1}{a^2}(ax-1)e^{ax}.$$

(b) Use the exponential Fourier series derived in part (a) to derive the corresponding trigonometric Fourier series.

Solution

(a) $f(t)$ has fundamental period $\tau = 2\pi$. Using equation (48), its exponential Fourier series takes the form

$$F(t) = \sum_{n=-\infty}^{\infty} C_n\,e^{int}.$$

From equation (49), the Fourier coefficients are given by

$$\begin{aligned} C_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-int}\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,e^{-int}\,dt. \end{aligned}$$

For $n = 0$, we have

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = 0$$

because the integrand is an odd function and the range of integration is symmetric about the origin.

For $n \neq 0$, the standard integral given in the question gives

$$\begin{aligned} C_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,e^{-int}\,dt \\ &= \frac{1}{2\pi(-in)^2}\left[(-int-1)\,e^{-int}\right]_{-\pi}^{\pi} \\ &= -\frac{1}{2\pi n^2}\left((-in\pi-1)\,e^{-in\pi} - (in\pi-1)\,e^{in\pi}\right). \end{aligned}$$

But

$$\begin{aligned} e^{\pm in\pi} &= \cos(n\pi) \pm i\sin(n\pi) \\ &= (-1)^n. \end{aligned}$$

So

$$C_n = -\frac{(-1)^n}{2\pi n^2}(-2in\pi)$$
$$= \frac{i}{n}(-1)^n.$$

The exponential Fourier series is therefore

$$F(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n}(-1)^n e^{int},$$

where the notation at the bottom of the summation symbol means that we sum n over all the integers except zero.

(b) Using equations (51), we get

$$A_0 = A_n = 0,$$
$$B_n = -2\,\mathrm{Im}(C_n) = -2\left[\frac{1}{n}(-1)^n\right] = \frac{2}{n}(-1)^{n+1}.$$

The corresponding trigonometric Fourier series is therefore

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nt).$$

This agrees with the trigonometric Fourier series found in Exercise 11.

Exercise 22

(a) Find the exponential Fourier series for the function

$$f(t) = e^{\alpha t} \quad \text{for } -1 \le t < 1,$$
$$f(t+2) = f(t),$$

where α is a real but non-zero number.

(b) Use your answer to part (a) to find the corresponding trigonometric Fourier series for $f(t)$.

The fast Fourier transform in the modern world

The fast Fourier transform (FFT) is an algorithm for calculating the exponential Fourier coefficients for a function, when the function is specified at discrete points only. As a result it is particularly suited to implementation on a computer. Its importance lies in its speed of calculation. Previous to its discovery, the calculation of N Fourier coefficients took around N^2 arithmetic operations.

However, in 1965, James Cooley and John Tukey announced the discovery of the FFT, which could calculate N Fourier coefficients in around $N \ln N$ operations. This improvement in speed is crucial when it is necessary to compute millions or billions of Fourier coefficients. Subsequently, in 1984, it was discovered that the algorithm was already known to the great mathematician Carl Friedrich Gauss as early as 1805 – pre-dating even Fourier's work.

The FFT is of huge importance. It is used ubiquitously in the mathematical and computational sciences, in topics from solving differential equations to algorithms for quick multiplication of large integers and matrices. It also has wider applications in the modern world, where it is used billions of times a day:

- for analysing and detecting signals
- for coding and decoding audio and speech signals, e.g. MP3 encoding
- for digital TV (DVB) and digital audio radio (DAB) broadcasting
- for background noise reduction in mobile telephony.

In fact, you couldn't log on to a Wi-Fi network or make a call on your mobile phone without the FFT. The FFT has rightly been described as 'the most important numerical algorithm of our lifetime' (Gilbert Strang).

4.2 An application to differential equations

Here we illustrate how an exponential Fourier series can be used to construct the solution of a differential equation.

As motivation, consider the following problem. If a tyre fails on the wheel of a vehicle (Figure 42), the wheel may no longer be circular, and the suspension of the vehicle will be subject to a periodic force, with period τ. The manufacturer wishes to estimate the effect of this periodic force on the body of the vehicle.

Figure 42 Can an aircraft survive landing on this damaged tyre? Fourier series can solve the relevant differential equation.

The motion of the body of the vehicle can be modelled by a damped, driven harmonic oscillator, of the type considered in Unit 3, with a periodic driving term. The differential equation that must be solved is

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = f(t), \tag{52}$$

In this model, x represents the vertical displacement of the body of the vehicle, relative to its equilibrium position. $f(t)$ depends on the properties of the burst tyre, and τ depends on the speed of the vehicle and the diameter of the tyre.

where the driving term $f(t)$ is a given periodic function, with fundamental period τ. Here, Γ is proportional to the damping constant, and ω_0 is the angular frequency of the harmonic oscillator in the absence of damping or external forces; this is called the **natural angular frequency**.

We can solve the differential equation in the usual way: first find the complementary function $x_c(t)$ that satisfies the auxiliary equation $\ddot{x}_c + 2\Gamma\dot{x}_c + \omega_0^2 x_c = 0$; then find a particular solution x; finally, construct the general solution $x_g = x + x_c$. Finding the complementary function is straightforward, but this term dies away as time increases, so the motion is eventually closely approximated by the particular solution, and that is what we concentrate on finding here.

Since the driving term $f(t)$ is a given periodic function, with fundamental period τ, we can compute its exponential Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \exp\left(\frac{2in\pi t}{\tau}\right) = \sum_{n=-\infty}^{\infty} f_n e^{in\omega t}, \tag{53}$$

where the f_n are the Fourier coefficients for $f(t)$ for $n = 1, 2, 3 \ldots$, and $\omega = 2\pi/\tau$ is the *angular frequency* of the driving force (not to be confused with the natural angular frequency of the system, ω_0).

Let us seek a particular solution $x(t)$ of equation (52) that is also a periodic function with period τ. Then we can also express $x(t)$ as a Fourier series, with Fourier coefficients x_n:

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{in\omega t}. \tag{54}$$

If we are able to determine the Fourier coefficients x_n, then equation (54) will give us the required particular solution of equation (52).

The differential equation involves the derivatives $\dot{x}$ and $\ddot{x}$, and we can get expressions for these by differentiating equation (54) term by term:

$$\dot{x}(t) = \sum_{n=-\infty}^{\infty} in\omega\, x_n e^{in\omega t}, \tag{55}$$

$$\ddot{x}(t) = \sum_{n=-\infty}^{\infty} (in\omega)^2 x_n e^{in\omega t} = \sum_{n=-\infty}^{\infty} (-n^2\omega^2) x_n e^{in\omega t}. \tag{56}$$

These are the exponential Fourier series for $\dot{x}$ and $\ddot{x}$.

Substituting equations (53)–(56) into equation (52), we obtain

$$\sum_{n=-\infty}^{\infty} (-n^2\omega^2 + 2\Gamma in\omega + \omega_0^2) x_n e^{in\omega t} = \sum_{n=-\infty}^{\infty} f_n e^{in\omega t}.$$

The sums on both sides of this equation are Fourier series. They can be equal only if all of their coefficients are equal, so

$$x_n = \frac{1}{(\omega_0^2 - n^2\omega^2) + 2\Gamma in\omega} f_n. \tag{57}$$

The particular solution that we seek is therefore

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{(\omega_0^2 - n^2\omega^2) + 2\Gamma in\omega} f_n e^{in\omega t}. \tag{58}$$

Using equations (51), this could also be expressed as a Fourier series involving sines and cosines, but we will not do this here.

For small values of the damping parameter, $\Gamma \simeq 0$, we get

$$x_n \simeq \frac{1}{\omega_0^2 - n^2\omega^2} f_n.$$

So if $f_n \neq 0$, the Fourier coefficient x_n can be very large when $\omega_0 \simeq n\omega$, for $n = 0, 1, 2, \ldots$, i.e. whenever the natural angular frequency is an integer multiple of the angular frequency of the driving force. This is a generalisation of the phenomenon of resonance discussed in Section 5 of Unit 3.

Example 15

Find a particular solution of the differential equation

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = f(t),$$

where the driving term $f(t)$ is the function discussed in Example 14.

Solution

From Example 14, we have $\tau = 2\pi$ so $\omega = 2\pi/\tau = 1$, and the exponential Fourier series for f is given by

$$F(t) = \sum_{n=-\infty}^{\infty} f_n e^{int},$$

where

$$f_0 = 0 \quad \text{and} \quad f_n = \frac{i}{n}(-1)^n \quad (n \neq 0).$$

From equation (57) we see that the differential equation has a particular solution

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int},$$

where

$$x_0 = 0 \quad \text{and} \quad x_n = \frac{i(-1)^n}{n} \, \frac{1}{(\omega_0^2 - n^2\omega^2) + 2\Gamma in\omega} \quad (n \neq 0).$$

The expression for x_n $(n \neq 0)$ can be simplified by the usual trick of multiplying the top and bottom by the complex conjugate of the bottom (see Unit 1). In the present case, the top and bottom are multiplied by $(\omega_0^2 - n^2\omega^2) - 2\Gamma in\omega$. Rearranging, and substituting the Fourier coefficients back into the exponential Fourier series, we conclude that

$$\begin{aligned} x(t) &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n} \, \frac{(\omega_0^2 - n^2\omega^2) - 2\Gamma in\omega}{(\omega_0^2 - n^2\omega^2)^2 + (2\Gamma in\omega)^2} \, e^{int} \\ &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} \, \frac{2\Gamma n\omega + i(\omega_0^2 - n^2\omega^2)}{(\omega_0^2 - n^2\omega^2)^2 - 4\Gamma^2 n^2\omega^2} \big(\cos(nt) + i\sin(nt)\big). \end{aligned}$$

It is easy to see that the imaginary terms in the sum are odd functions of n. The imaginary part of $x(t)$ is therefore equal to 0 because the contributions with $n > 0$ exactly cancel those with $n < 0$.

Learning outcomes

After studying this unit, you should be able to do the following.

- Understand the terms angular frequency, period, fundamental period and fundamental interval, and be able to obtain their values for a periodic function.
- Give correct specifications of piecewise defined functions.
- Determine whether a function is even or odd, and use evenness and oddness to simplify definite integrals over a range that is symmetric about the origin.
- Calculate the trigonometric Fourier series for any periodic function, taking advantage of evenness or oddness where appropriate.
- Understand how to modify a function defined on a finite interval to give its even or odd periodic extension, and hence represent the function by a trigonometric Fourier series.
- Calculate the value of a Fourier series at a point of discontinuity.
- Where appropriate, simplify the calculation of Fourier coefficients by shifting the range of integration.
- Differentiate Fourier series to obtain new Fourier series from old.

Appendix: proofs and orthogonality

This Appendix is optional and will not be assessed. It justifies the equations used to calculate Fourier coefficients and, more importantly, casts further light on the analogy between Fourier series and vectors. This analogy is frequently used in areas of advanced mathematics and in subjects like quantum mechanics.

Formulas for trigonometric Fourier coefficients

Given a function $f(t)$ with fundamental period τ, its trigonometric Fourier series is defined to be

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{\tau}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi t}{\tau}\right), \qquad (59)$$

where A_0, A_n and B_n are the Fourier coefficients for $f(t)$ for $n = 1, 2, 3, \ldots$. This series is equal to the original function $f(t)$ (except at isolated points where $f(t)$ is discontinuous). So we write

$$f(t) = F(t).$$

The main text showed how to calculate A_0 by integrating both sides of equation (59) from $-\tau/2$ to $\tau/2$. The integrals of the sine functions vanish because their integrands are odd. The integrals of the cosine functions also vanish because we are integrating over a whole number of their periods. Explicitly, if $n \neq 0$ is an integer,

$$\begin{aligned}\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right) dt &= \left[\frac{\tau}{2n\pi}\sin\left(\frac{2n\pi t}{\tau}\right)\right]_{-\tau/2}^{\tau/2} \\ &= \frac{\tau}{2n\pi}\big(\sin(n\pi) - \sin(-n\pi)\big) = 0. \qquad (60)\end{aligned}$$

Hence integration of equation (59) from $-\tau/2$ to $\tau/2$ gives

$$\int_{-\tau/2}^{\tau/2} F(t)\, dt = A_0\tau.$$

Using $f(t) = F(t)$, we obtain the following expression for A_0:

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, dt.$$

The main text gave only a sketch of the derivation of the remaining Fourier coefficients, indicating how A_2 can be found. We now give more details. The key idea is as follows: we multiply both sides of equation (59) by a suitable function, chosen so that when we integrate from $-\tau/2$ to $\tau/2$, the integrals of all the terms vanish, except one. This remaining non-zero integral is multiplied by the Fourier coefficient that we want to find, and a simple rearrangement then gives a formula for that coefficient.

The calculation of A_0 is a trivial example of this idea, in which the multiplying function is the unit function, 1.

To implement this broad plan of action, we will use the following set of standard integrals, where n and m are any positive integers:

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right)\sin\left(\frac{2m\pi t}{\tau}\right) dt = 0, \qquad (61)$$

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right)\cos\left(\frac{2m\pi t}{\tau}\right) dt = \begin{cases}\tau/2 & \text{for } n = m,\\ 0 & \text{for } n \neq m,\end{cases} \qquad (62)$$

$$\int_{-\tau/2}^{\tau/2} \sin\left(\frac{2n\pi t}{\tau}\right)\sin\left(\frac{2m\pi t}{\tau}\right) dt = \begin{cases}\tau/2 & \text{for } n = m,\\ 0 & \text{for } n \neq m.\end{cases} \qquad (63)$$

Equation (61) is obviously true. The sine function is odd and the cosine function is even, so their product is an odd function. This is integrated over a range that is symmetric about the origin, so the integral is equal to zero.

To establish equation (62), we use the trigonometric identity

$$\cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)],$$

which gives

$$\cos\left(\frac{2n\pi t}{\tau}\right)\cos\left(\frac{2m\pi t}{\tau}\right) = \frac{1}{2}\left[\cos\left(\frac{2(n-m)\pi t}{\tau}\right) + \cos\left(\frac{2(n+m)\pi t}{\tau}\right)\right].$$

Let us assume for the moment that $n \neq m$. Then, since n and m are positive integers, $n - m$ and $n + m$ are non-zero integers. We can therefore

use the result of equation (60) (with n replaced by $n \pm m$) to show that the integral in equation (62) is equal to zero when $n \neq m$. This leaves the case $n = m$, for which the above trigonometric identity gives

$$\cos\left(\frac{2n\pi t}{\tau}\right)\cos\left(\frac{2n\pi t}{\tau}\right) = \frac{1}{2}\left[1 + \cos\left(\frac{4n\pi t}{\tau}\right)\right].$$

Again, the integral from $-\tau/2$ to $\tau/2$ of the cosine term gives zero, and we are left with

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right)\cos\left(\frac{2n\pi t}{\tau}\right) dt = \int_{-\tau/2}^{\tau/2} \frac{1}{2}\, dt = \frac{\tau}{2}.$$

This completes the proof of equation (62).

Equation (63) is proved in a similar way, using the trigonometric identity

$$\sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)],$$

which gives

$$\sin\left(\frac{2n\pi t}{\tau}\right)\sin\left(\frac{2m\pi t}{\tau}\right) = \frac{1}{2}\left[\cos\left(\frac{2(n-m)\pi t}{\tau}\right) - \cos\left(\frac{2(n+m)\pi t}{\tau}\right)\right].$$

For exactly the same reasons as above, the integrals from $-\tau/2$ to $\tau/2$ of these terms are equal to zero provided that $n \neq m$. For the special case $n = m$ we have

$$\sin\left(\frac{2n\pi t}{\tau}\right)\sin\left(\frac{2n\pi t}{\tau}\right) = \frac{1}{2}\left[1 - \cos\left(\frac{4n\pi t}{\tau}\right)\right].$$

The integral from $-\tau/2$ to $\tau/2$ of the cosine term again gives zero, so

$$\int_{-\tau/2}^{\tau/2} \sin\left(\frac{2n\pi t}{\tau}\right)\sin\left(\frac{2n\pi t}{\tau}\right) dt = \int_{-\tau/2}^{\tau/2} \frac{1}{2}\, dt = \frac{\tau}{2},$$

completing the proof of equation (63).

Equations (62) and (63) can be tidied up using a shorthand notation that is surprisingly useful. The **Kronecker delta symbol** is defined by

The Kronecker delta symbol is named after the German mathematician Leopold Kronecker (1823–1891).

$$\delta_{nm} = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases}$$

Using this notation, the integrals that we need can be expressed as follows.

Integrals needed to find Fourier coefficients

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right)\sin\left(\frac{2m\pi t}{\tau}\right) dt = 0, \tag{64}$$

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right)\cos\left(\frac{2m\pi t}{\tau}\right) dt = \frac{\tau}{2}\,\delta_{nm}, \tag{65}$$

$$\int_{-\tau/2}^{\tau/2} \sin\left(\frac{2n\pi t}{\tau}\right)\sin\left(\frac{2m\pi t}{\tau}\right) dt = \frac{\tau}{2}\,\delta_{nm}. \tag{66}$$

Using these integrals, we can easily obtain formulas for the Fourier coefficients A_n and B_n for $n = 1, 2, 3, \ldots$.

Let us multiply both sides of equation (59) by $\cos(2m\pi t/\tau)$, where m is a positive integer, and then integrate from $-\tau/2$ to $\tau/2$. Integrating each term in turn and using equations (64) and (65), we get

$$\int_{-\tau/2}^{\tau/2} F(t) \cos\left(\frac{2m\pi t}{\tau}\right) dt = \sum_{n=1}^{\infty} A_n \, \frac{\tau}{2} \, \delta_{nm} + 0.$$

In the sum on the right-hand side, the Kronecker delta symbol ensures that practically all the terms are equal to zero. Only the term with $n = m$ survives. Using the fact that $F(t) = f(t)$, we therefore conclude that

$$A_m = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos\left(\frac{2m\pi t}{\tau}\right) dt \quad (m = 1, 2, 3, \ldots). \tag{67}$$

Similarly, multiplying both sides of equation (59) by $\sin(2m\pi t/\tau)$, where m is a positive integer, and then integrating from $-\tau/2$ to $\tau/2$ gives

$$\int_{-\tau/2}^{\tau/2} F(t) \sin\left(\frac{2m\pi t}{\tau}\right) dt = 0 + \sum_{n=1}^{\infty} B_n \, \frac{\tau}{2} \, \delta_{nm} = \frac{\tau}{2} B_m.$$

So, using the fact that $F(t) = f(t)$, we get

$$B_m = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \sin\left(\frac{2m\pi t}{\tau}\right) dt \quad (m = 1, 2, 3, \ldots). \tag{68}$$

Equations (67) and (68) give the Fourier coefficients A_m and B_m for $m = 1, 2, 3, \ldots$. We may, of course, replace the label m by n to recover equations (19) and (20).

Formulas for exponential Fourier coefficients

Very similar arguments apply to the exponential Fourier series for a function $f(t)$ with fundamental period τ. In this case, we define

$$F(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{2in\pi t/\tau}. \tag{69}$$

This Fourier series is equal to the original function $f(t)$ (except at isolated points where $f(t)$ is discontinuous). So we can write

$$f(t) = F(t).$$

We can again 'pick off' the Fourier coefficients by multiplying both sides by a suitable function and integrating. In this case, the standard integral that unlocks the coefficients is as follows.

$$\int_{-\tau/2}^{\tau/2} e^{-2im\pi t/\tau} e^{2in\pi t/\tau} \, dt = \tau \, \delta_{nm}, \tag{70}$$

where n and m are integers and δ_{nm} is the Kronecker delta symbol.

This integral is quite easy to establish. Assuming that $n \neq m$ are integers, we get

$$\begin{aligned}\int_{-\tau/2}^{\tau/2} e^{-2im\pi t/\tau} e^{2in\pi t/\tau}\,dt &= \int_{-\tau/2}^{\tau/2} e^{2i(n-m)\pi t/\tau}\,dt \\ &= \frac{\tau}{2i(n-m)\pi}\Big[e^{2i(n-m)\pi t/\tau}\Big]_{-\tau/2}^{\tau/2} \\ &= \frac{\tau}{2i(n-m)\pi}\Big(e^{i(n-m)\pi} - e^{-i(n-m)\pi}\Big) \\ &= \frac{\tau}{(n-m)\pi}\sin((n-m)\pi) = 0.\end{aligned}$$

Recall that $\sin x = \dfrac{e^x - e^{-x}}{2i}$.

On the other hand, if $n = m$, the integrand becomes equal to 1, so

$$\int_{-\tau/2}^{\tau/2} e^{-2in\pi t/\tau} e^{2in\pi t/\tau}\,dt = \int_{-\tau/2}^{\tau/2} 1\,dt = \tau,$$

which completes the proof of equation (70).

Given equation (70), it is easy to derive a formula for the Fourier coefficients. We just multiply both sides of equation (69) by the factor $e^{-2im\pi t/\tau}$ and integrate term by term from $-\tau/2$ to $\tau/2$. This gives

$$\int_{-\tau/2}^{\tau/2} F(t)\, e^{-2im\pi t/\tau}\,dt = \sum_{n=-\infty}^{\infty} C_n\, \tau\, \delta_{nm}.$$

Again, the Kronecker delta symbol ensures that the sum on the right-hand side reduces to the single term $C_m\tau$. Hence, using the fact that $F(t) = f(t)$, we get

$$C_m = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\, e^{-2im\pi t/\tau}\,dt.$$

Replacing the arbitrary label m by n, we then recover equation (49).

Orthogonal functions

Finally, let us return to the analogy between Fourier series and vectors mentioned earlier in the text.

You know that a vector in three-dimensional space can be expressed as

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}, \tag{71}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are Cartesian unit vectors, and v_x, v_y and v_z are the corresponding components of $\mathbf{v}$. A key property of these unit vectors is that they are mutually orthogonal. This means that their scalar products satisfy

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0. \tag{72}$$

The unit vectors also have unit magnitude, so

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1. \tag{73}$$

These properties allow us to isolate any one of the components of $\mathbf{v}$. For example, taking the scalar product of equation (71) with $\mathbf{i}$ and using equations (72) and (73), we get

$$\begin{aligned}\mathbf{i}\cdot\mathbf{v} &= v_x\,(\mathbf{i}\cdot\mathbf{i}) + v_y\,(\mathbf{i}\cdot\mathbf{j}) + v_z\,(\mathbf{i}\cdot\mathbf{k})\\ &= v_x\,(1) + v_y\,(0) + v_z\,(0)\\ &= v_x.\end{aligned}$$

Note that taking the scalar product of $\mathbf{i}$ with $\mathbf{v}$ is achieved by taking the scalar product of $\mathbf{i}$ with each of the terms on the right-hand side of equation (71). Technically, we say that the scalar product is distributive. The component v_x can then be isolated because the unit vectors are orthogonal.

Similar language can be used to describe the calculation of Fourier coefficients. In this case, equation (71) for a vector is replaced by the Fourier series for a periodic function (equation (59) or (69)). Instead of taking the scalar product with a unit vector, we multiply by a suitable function and integrate over the fundamental period τ. The process of integration is also distributive, meaning that we can integrate each term on the right-hand side of the Fourier series in turn. Corresponding to the orthogonality of the unit vectors, we then have the standard integrals in equations (64)–(66) and (70):

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right) \sin\left(\frac{2m\pi t}{\tau}\right) dt = 0,$$

$$\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2n\pi t}{\tau}\right) \cos\left(\frac{2m\pi t}{\tau}\right) dt = \frac{\tau}{2}\,\delta_{nm},$$

$$\int_{-\tau/2}^{\tau/2} \sin\left(\frac{2n\pi t}{\tau}\right) \sin\left(\frac{2m\pi t}{\tau}\right) dt = \frac{\tau}{2}\,\delta_{nm},$$

$$\int_{-\tau/2}^{\tau/2} e^{-2im\pi t/\tau} e^{2in\pi t/\tau}\, dt = \tau\,\delta_{nm}.$$

This is what allows us to pick off individual Fourier coefficients from the sum. Because of these analogies, we say that the functions

$$\cos\left(\frac{2n\pi t}{\tau}\right),\quad \cos\left(\frac{4n\pi t}{\tau}\right),\quad \cos\left(\frac{6n\pi t}{\tau}\right),\quad \ldots,$$

$$\sin\left(\frac{2n\pi t}{\tau}\right),\quad \sin\left(\frac{4n\pi t}{\tau}\right),\quad \sin\left(\frac{6n\pi t}{\tau}\right),\quad \ldots,$$

and the constant function 1 are **orthogonal functions**. Similarly, the functions

$$e^{2in\pi t/\tau},\quad e^{4in\pi t/\tau},\quad e^{6in\pi t/\tau},\quad \ldots$$

are said to be orthogonal. In both cases, orthogonality refers to the fact that certain integrals of these functions (given in equations (64)–(66) and (70)) are equal to zero. These integrals are sometimes called **orthogonality integrals**.

A minor difference compared with vectors is that the orthogonality integrals do not give 1 when $n = m$, so we do not have an analogue for equation (73). This could easily be remedied by scaling the functions, but this is not needed for our purposes. The important point to take away is that you can visualise the process of calculating Fourier coefficients as being similar to the process of projecting a vector onto the coordinate axes to find its components.

Solutions to exercises

Solution to Exercise 1

(a) The function $\sin\left(\frac{1}{4}x\right)$ has angular frequency $\omega = \frac{1}{4}$, so its fundamental period is $\tau = 2\pi/\omega = 8\pi$.

(b) The function $\cos\left(\frac{2}{5}x\right)$ has angular frequency $\omega = \frac{2}{5}$, so its fundamental period is $\tau = 2\pi/\omega = 5\pi$.

(c) The function $\sin\left(\frac{1}{4}x\right) + 2\cos\left(\frac{2}{5}x\right)$ is the sum of two functions: $\sin\left(\frac{1}{4}x\right)$ and $2\cos\left(\frac{2}{5}x\right)$. The complete set of periods for $\sin\left(\frac{1}{4}x\right)$ is given by the positive integer multiples of 8π, that is,

$$8\pi,\ 16\pi,\ 24\pi,\ 32\pi,\ 40\pi,\ 48\pi,\ \ldots.$$

The complete set of periods for $2\cos\left(\frac{2}{5}x\right)$ is given by the positive integer multiples of 5π, that is,

$$5\pi,\ 10\pi,\ 15\pi,\ 20\pi,\ 25\pi,\ 30\pi,\ 35\pi,\ 40\pi,\ \ldots.$$

The smallest period that these functions have in common is 40π. This is the fundamental period of $\sin\left(\frac{1}{4}x\right) + 2\cos\left(\frac{2}{5}x\right)$.

Solution to Exercise 2

The given function has period 3, so $q(t+3) = q(t)$. Hence

$$q(1000) = q(1 + 3 \times 333) = q(1) = 1,$$

where the final value is obtained by examination of Figure 4. Similarly,

$$q(-77) = q(1 - 78) = q(1 - 3 \times 26) = q(1) = 1.$$

Solution to Exercise 3

Looking at Figure 11, we see that $q(t) = -\frac{1}{2}t$ for $-2 \le t \le 0$. Also, $q(t) = t$ for $0 < t \le 1$, and $q(t) = \frac{3}{2} - \frac{1}{2}t$ for $1 < t \le 3$.

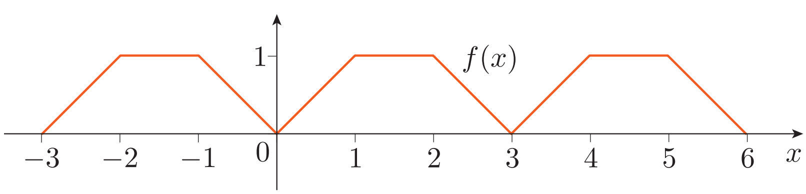
Hence over the fundamental interval $-\frac{3}{2} \le t \le \frac{3}{2}$, $q(t)$ can be defined by

$$q(t) = \begin{cases} -\frac{1}{2}t & \text{for } -\frac{3}{2} \le t \le 0, \\ t & \text{for } 0 < t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le \frac{3}{2}, \end{cases}$$

$$q(t+3) = q(t).$$

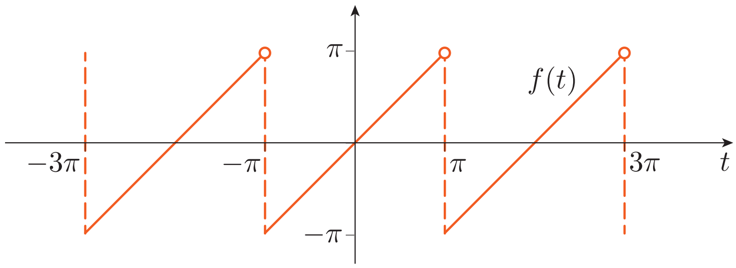
Solution to Exercise 4

A sketch of the function is shown below.



Solution to Exercise 5

This function is shown in the figure below. Note the positions of the open circles.



Solution to Exercise 6

(a) $u(t)$ has discontinuities at $t = \pm 1, \pm 3, \pm 5, \ldots$, i.e. at $t = 2n + 1$ for n an integer.

The fundamental period is $\tau = 2$.

(b) The piecewise definition is

$$u(t) = \begin{cases} t+1 & \text{for } -1 \leq t \leq 0, \\ 1 & \text{for } 0 < t < 1, \end{cases}$$

$$u(t+2) = u(t).$$

(c) Since $u(t+2n) = u(t)$ for n an integer, $u(99) = u(-1) = 0$ and $u(100) = u(0) = 1$.

Solution to Exercise 7

(a) If $f(x) = x^3 - 3x$, then

$$f(-x) = (-x)^3 - 3(-x) = -(x^3 - 3x) = -f(x),$$

so this function is odd.

(b) The functions $\sin x$ and $\sin(4x)$ are both odd, so if $f(x) = 2\sin x + 3\sin(4x)$, then $f(x)$ is an odd function.

(c) Each term in the sum $5 + 2\cos x + 7\cos(4x)$ is even, so this function is even.

(d) The constant function 4 is even and $\sin x$ is odd, so if $f(x) = 4 - 2\sin x$, then $f(-x) = 4 + 2\sin x$. So $f(x)$ is neither even nor odd.

(e) If $f(x) = 2x\cos(3x)$, then

$$f(-x) = -2x\cos(-3x) = -2x\cos(3x) = -f(x),$$

so this function is odd.

Solution to Exercise 8

(a) The function x^3 is odd and the function $\cos(2x)$ is even, so the integrand is odd. The range of integration is symmetric about the origin, so the integral vanishes, i.e.

$$\int_{-1}^{1} x^3 \cos(2x)\,dx = 0.$$

(b) The function $\sin(2x^3)$ is odd and the constant function 3 is even. Since the range of integration is symmetric about the origin, we have

$$\begin{aligned}\int_{-2}^{2} (3+\sin(2x^3))\,dx &= \int_{-2}^{2} 3\,dx \\ &= 6\int_0^2 1\,dx = \big[6x\big]_0^2 = 12.\end{aligned}$$

We have used the fact that a constant is an even function, but this is an optional step.

Solution to Exercise 9

Looking at Figure 3, we see that $p(t)$ has fundamental period $\tau = 2$, therefore from equation (10), its Fourier series has the form

$$\begin{aligned}P(t) &= A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi t) + \sum_{n=1}^{\infty} B_n \sin(n\pi t) \\ &= A_0 + \big(A_1\cos(\pi t) + A_2\cos(2\pi t) + \cdots\big) \\ &\quad + \big(B_1 \sin(\pi t) + B_2\sin(2\pi t) + \cdots\big).\end{aligned}$$

Comparing with equation (2), we see that $A_0 = A_1 = A_2 = A_3 = A_4 = 0$ and

$$B_1 = \frac{4}{\pi^2}, \quad B_2 = 0, \quad B_3 = -\frac{4}{9\pi^2}, \quad B_4 = 0.$$

Solution to Exercise 10

From Figure 2, it is clear that $|\cos t|$ has period $\tau = \pi$. Hence using equation (14) we have

$$A_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} c(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |\cos t|\,dt = \frac{2}{\pi}\int_0^{\pi/2} |\cos t|\,dt,$$

where the last step follows because $|\cos t|$ is an even function.

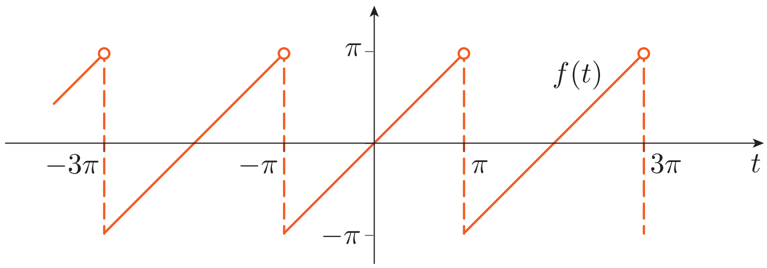
But over the range $-\pi/2 \le t \le \pi/2$, $\cos t$ is positive, so $|\cos t| = \cos t$. Hence

$$\begin{aligned}A_0 &= \frac{2}{\pi}\int_0^{\pi/2} \cos t\,dt \\ &= \frac{2}{\pi}\big[\sin t\big]_0^{\pi/2} \\ &= \frac{2}{\pi}\big(\sin(\pi/2) - \sin(0)\big) = \frac{2}{\pi},\end{aligned}$$

in agreement with the stated Fourier series in equation (1).

Solution to Exercise 11

This function is sketched in the figure below.



$f(t)$ is an odd function. It has fundamental period $\tau = 2\pi$, so its Fourier series (equation (17)) takes the form

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nt) + \sum_{n=1}^{\infty} B_n \sin(nt).$$

From equation (18) we get

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt = 0,$$

because $f(t)$ is odd and the range of integration is symmetric about the origin.

From equation (19) we get

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt \quad (n = 1, 2, 3, \ldots).$$

Because $f(t)$ is odd and $\cos(nt)$ is even, the integrand $f(t)\cos(nt)$ is an odd function, so when this is integrated from $t = -\pi$ to $t = \pi$, we get zero. Hence

$$A_n = 0 \quad (n = 1, 2, 3, \ldots).$$

Finally, from equation (20) we get

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin(nt)\, dt.$$

Using the standard integral in equation (32), we get

$$\begin{aligned} B_n &= \frac{1}{n^2\pi} \big[\sin(nt) - nt\cos(nt)\big]_{-\pi}^{\pi} \\ &= \frac{1}{n^2\pi} \big(\sin(n\pi) - n\pi\cos(n\pi) - (\sin(-n\pi) + n\pi\cos(-n\pi))\big). \end{aligned}$$

Since $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ for n an integer, this simplifies to

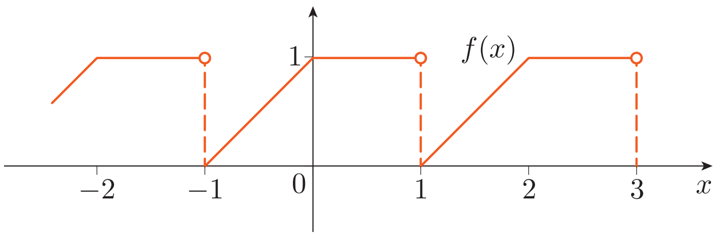
$$B_n = \frac{1}{n^2\pi}(-2n\pi(-1)^n) = \frac{2}{n}(-1)^{n+1} \quad (n = 1, 2, 3, \ldots).$$

Putting these results together, the required Fourier series is

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nt).$$

Solution to Exercise 12

The function is sketched in the figure below.



This function is neither even nor odd. It has fundamental period $\tau = 2$, so its Fourier series (equation (17)) has the form

$$F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) + \sum_{n=1}^{\infty} B_n \sin(n\pi x).$$

From equation (18) we get

$$A_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx.$$

To integrate a function defined on pieces, we integrate each piece in turn using the correct value of the function on each piece. So

$$\begin{aligned} A_0 &= \tfrac{1}{2}\left(\int_{-1}^{0}(x+1)\,dx + \int_{0}^{1} 1\,dx\right) \\ &= \tfrac{1}{2}\left(\left[\tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \left[x\right]_{0}^{1}\right) = \tfrac{1}{2}\left(-(\tfrac{1}{2}-1)+1\right) = \tfrac{3}{4}. \end{aligned}$$

Similarly, equation (19) gives

$$A_n = \frac{2}{2}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx \quad (n = 1, 2, 3, \ldots),$$

so

$$A_n = \int_{-1}^{0}(x+1)\cos(n\pi x)\,dx + \int_{0}^{1}\cos(n\pi x)\,dx.$$

For the integrals of $\cos(n\pi x)$, we can join the two integration ranges together. Hence

$$A_n = \int_{-1}^{0} x\cos(n\pi x)\,dx + \int_{-1}^{1}\cos(n\pi x)\,dx.$$

Using the standard integral in equation (33), we then get

$$\begin{aligned} A_n &= \left[\frac{1}{(n\pi)^2}\big(\cos(n\pi x) + n\pi x\sin(n\pi x)\big)\right]_{-1}^{0} + \left[\frac{1}{n\pi}\sin(n\pi x)\right]_{-1}^{1} \\ &= \frac{1}{(n\pi)^2}\big(\cos(0) - \cos(-n\pi)\big) \\ &= \frac{1}{(n\pi)^2}\left(1 - (-1)^n\right) \quad (n = 1, 2, 3, \ldots), \end{aligned}$$

where we have used $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ for n an integer.

Also, equation (20) gives

$$\begin{aligned}
B_n &= \frac{2}{2}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx \\
&= \int_{-1}^{0}(x+1)\sin(n\pi x)\,dx + \int_{0}^{1}\sin(n\pi x)\,dx \\
&= \int_{-1}^{0} x\sin(n\pi x)\,dx + \int_{-1}^{1}\sin(n\pi x)\,dx.
\end{aligned}$$

Using the standard integral in equation (32), we get

$$B_n = \left[\frac{1}{(n\pi)^2}\big(\sin(n\pi x) - n\pi x\cos(n\pi x)\big)\right]_{-1}^{0} - \left[\frac{1}{n\pi}\cos(n\pi x)\right]_{-1}^{1}.$$

Since $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ for n an integer, we get

$$\begin{aligned}
B_n &= \frac{1}{(n\pi)^2}(0-0) - \frac{1}{(n\pi)^2}\big(\sin(-n\pi) + n\pi\cos(-n\pi)\big) \\
&\quad - \frac{1}{n\pi}\big(\cos(n\pi) - \cos(-n\pi)\big) \\
&= -\frac{1}{n\pi}(-1)^n \quad (n = 1, 2, 3, \ldots).
\end{aligned}$$

Putting all these results together, the required Fourier series is

$$F(x) = \frac{3}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(1-(-1)^n)\cos(n\pi x) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x).$$

The first few A_n are

$$A_1 = \frac{2}{1^2\pi^2}, \quad A_2 = 0, \quad A_3 = \frac{2}{3^2\pi^2}, \quad A_4 = 0, \quad A_5 = \frac{2}{5^2\pi^2}.$$

Hence the Fourier series can also be written as

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\pi x) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x).$$

Solution to Exercise 13

It is clear from the figure given in the question that $f(t)$ is an even function with fundamental period $\tau = 2$. Following Procedure 3, its Fourier series involves only constant and cosine terms, and takes the form

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos(n\pi t).$$

The constant term is

$$A_0 = \frac{2}{2}\int_0^1 f(t)\,dt = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}.$$

The remaining Fourier coefficients are

$$A_n = \frac{4}{2}\int_0^1 f(t)\cos(n\pi t)\,dt = 2\int_0^1 t\cos(n\pi t)\,dt.$$

This integral can be evaluated using equation (33). We get

$$\begin{aligned} A_n &= \frac{2}{(n\pi)^2}\big[\cos(n\pi t) + n\pi t\sin(n\pi t)\big]_0^1 \\ &= \frac{2}{(n\pi)^2}\big(\cos(n\pi) - \cos(0)\big) \\ &= \frac{2}{(n\pi)^2}\big((-1)^n - 1\big) \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

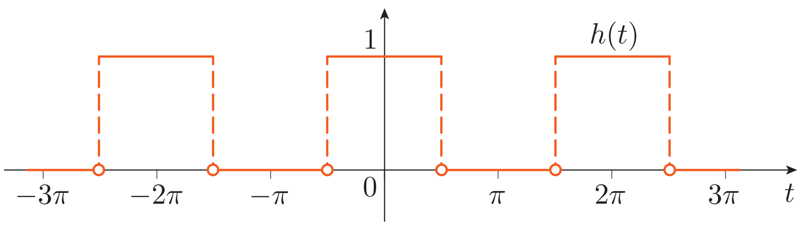
The Fourier series is therefore

$$F(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\big((-1)^n - 1\big)\cos(n\pi t),$$

which agrees with the answer given in Example 7.

Solution to Exercise 14

The function $h(t)$ is sketched in the figure below.



Clearly $h(t)$ is even and has fundamental period $\tau = 2\pi$. Following Procedure 3, its Fourier series involves only constant and cosine terms, and takes the form

$$H(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nt).$$

The Fourier coefficients are evaluated as follows:

$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_0^{\pi} h(t)\,dt \\ &= \frac{1}{\pi}\int_0^{\pi/2} 1\,dt + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\,dt \\ &= \frac{1}{\pi}\big[t\big]_0^{\pi/2} = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} h(t)\cos(nt)\,dt \\ &= \frac{2}{\pi}\int_0^{\pi/2} \cos(nt)\,dt \\ &= \frac{2}{n\pi}\big[\sin(nt)\big]_0^{\pi/2} \\ &= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

Hence the Fourier series is given by

$$H(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nt).$$

For $n = 1, 2, 3, 4, 5$, the values of $\sin(n\pi/2)$ are $1, 0, -1, 0, 1$, so the first few terms of the Fourier series are

$$H(t) = \frac{1}{2} + \frac{2}{\pi}\cos(t) - \frac{2}{3\pi}\cos(3t) + \frac{2}{5\pi}\cos(5t) - \cdots,$$

in agreement with the Fourier series stated in equation (7).

Only the odd values of n contribute to the Fourier series. Putting $n = 2m - 1$ and noting that equation (24) gives

$$\sin((2m-1)\pi/2) = (-1)^{m+1},$$

this Fourier series can also be written in the alternative form

$$H(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1} \cos((2m-1)t).$$

Solution to Exercise 15

It is clear from Figure 2 that $c(t)$ is even and has fundamental period $\tau = \pi$. Following Procedure 3, its Fourier series involves only constant and cosine terms, and takes the form

$$C(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2nt).$$

On the interval $0 \le t \le \pi/2$, we have $c(t) = \cos t$. Therefore

$$\begin{aligned} A_0 &= \frac{2}{\pi} \int_0^{\pi/2} |\cos t| \, dt \\ &= \frac{2}{\pi} \int_0^{\pi/2} \cos t \, dt \\ &= \frac{2}{\pi} \big[\sin t\big]_0^{\pi/2} = \frac{2}{\pi}. \end{aligned}$$

Similarly,

$$A_n = \frac{4}{\pi} \int_0^{\pi/2} \cos t \cos(2nt) \, dt.$$

Using the standard integral given in the question, with $a = 1$ and $b = 2n$, we then get

$$\begin{aligned} A_n &= \frac{4}{\pi} \left[\frac{2n \cos t \sin(2nt) - \cos(2nt) \sin t}{4n^2 - 1} \right]_0^{\pi/2} \\ &= \frac{4}{\pi} \left(\frac{2n \cos(\pi/2) \sin(n\pi) - \cos(n\pi) \sin(\pi/2)}{4n^2 - 1} \right) \\ &= -\frac{4}{\pi} \frac{1}{4n^2 - 1} \cos(n\pi). \end{aligned}$$

But $\cos(n\pi) = (-1)^n$, so

$$A_n = \frac{4}{\pi}\frac{(-1)^{n+1}}{4n^2-1} \quad (n = 1, 2, 3, \ldots).$$

Thus the Fourier series is

$$C(t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos(2nt).$$

The first few terms of this Fourier series are

$$C(t) = \frac{4}{\pi}\left(\tfrac{1}{2} + \tfrac{1}{3}\cos(2t) - \tfrac{1}{15}\cos(4t) + \tfrac{1}{35}\cos(6t) - \cdots\right),$$

in agreement with equation (1) in the Introduction.

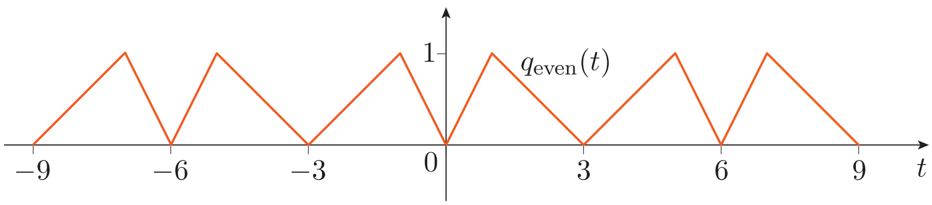
Solution to Exercise 16

Using the definition of the even periodic extension, we have

$$q_{\text{even}}(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le 3, \\ -t & \text{for } -1 \le t < 0, \\ \frac{3}{2} + \frac{1}{2}t & \text{for } -3 < t < -1, \end{cases}$$

$$q_{\text{even}}(t+6) = q_{\text{even}}(t).$$

This function has fundamental period $\tau = 6$, and its formula cannot be made much simpler. Its graph is sketched below.

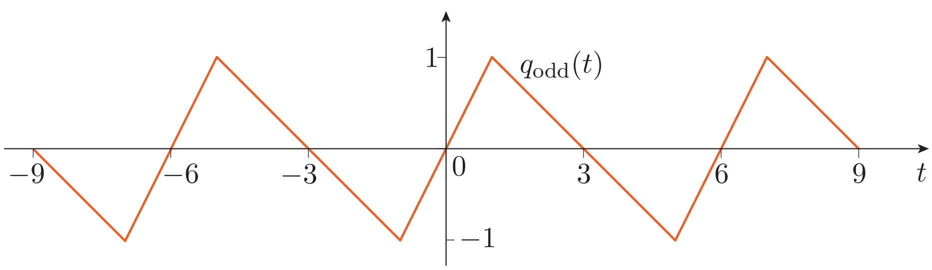


The odd periodic extension is given by

$$q_{\text{odd}}(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le 3, \\ t & \text{for } -1 \le t < 0, \\ -\frac{3}{2} - \frac{1}{2}t & \text{for } -3 < t < -1, \end{cases}$$

$$q_{\text{odd}}(t+6) = q_{\text{odd}}(t).$$

This function has fundamental period $\tau = 6$, and its graph is sketched below.



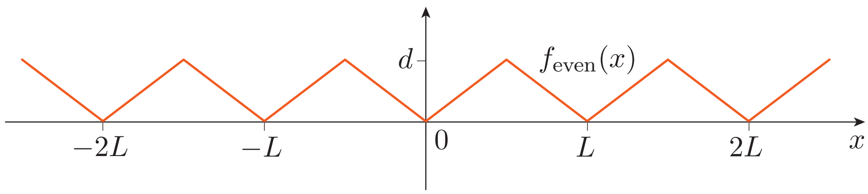
By examining this graph, we see that the formula can be simplified to

$$q_{\text{odd}}(t) = \begin{cases} t & \text{for } -1 < t \le 1, \\ \frac{3}{2} - \frac{1}{2}t & \text{for } 1 < t \le 5, \end{cases}$$

$$q_{\text{odd}}(t+6) = q_{\text{odd}}(t).$$

Solution to Exercise 17

Because we are looking for a Fourier series that involves only constant and cosine terms, we need to consider the *even* periodic extension of $f(x)$, denoted by $f_{\text{even}}(x)$. This is sketched in the figure below.



The function $f_{\text{even}}(x)$ is even, and from the above sketch has period $\tau = L$, so its Fourier series takes the form

$$F_{\text{even}}(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi x}{L}\right),$$

where the Fourier coefficients A_0 and A_n are given by

$$A_0 = \frac{2}{L}\int_0^{L/2} f_{\text{even}}(x)\,dx,$$

$$A_n = \frac{4}{L}\int_0^{L/2} f_{\text{even}}(x)\cos\left(\frac{2n\pi x}{L}\right)dx \quad (n = 1, 2, 3, \ldots).$$

But on the interval $0 \le x \le L/2$,

$$f_{\text{even}}(x) = f(x) = \frac{2d}{L}x.$$

Hence

$$A_0 = \frac{4d}{L^2}\int_0^{L/2} x\,dx = \frac{4d}{L^2}\left[\frac{x^2}{2}\right]_0^{L/2} = \frac{d}{2}$$

and

$$A_n = \frac{8d}{L^2}\int_0^{L/2} x\cos\left(\frac{2n\pi x}{L}\right)dx \quad (n = 1, 2, 3, \ldots).$$

Using equation (33), we get

$$\begin{aligned} A_n &= \frac{8d}{L^2}\left(\frac{L}{2n\pi}\right)^2\left[\cos\left(\frac{2n\pi x}{L}\right) + \frac{2n\pi x}{L}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^{L/2} \\ &= \frac{2d}{n^2\pi^2}\big(\cos(n\pi) - \cos(0)\big) \\ &= \frac{2d}{n^2\pi^2}\big((-1)^n - 1\big) \quad (n = 1, 2, 3, \ldots). \end{aligned}$$

So

$$F_{\text{even}}(x) = \frac{d}{2} + \frac{2d}{\pi^2} \sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{n^2} \cos\left(\frac{2n\pi x}{L}\right).$$

Since $f(x)$ and $f_{\text{even}}(x)$ coincide on the interval $0 \le x \le L$, this is the required cosine Fourier series $F(x)$ for $f(x)$.

The first few terms in the Fourier series are

$$F(x) = \frac{d}{2} - \frac{4d}{\pi^2}\left[\cos\left(\frac{2\pi x}{L}\right) + \frac{1}{3^2}\cos\left(\frac{6\pi x}{L}\right) + \frac{1}{5^2}\cos\left(\frac{10\pi x}{L}\right) + \cdots\right].$$

This agrees with the result of Exercise 13 in the special case $d = 1$, $L = 2$.

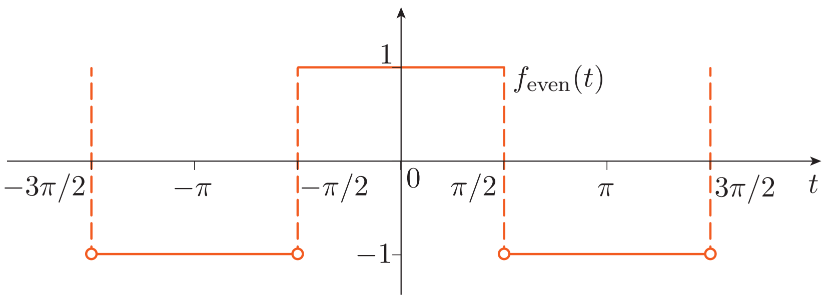
Solution to Exercise 18

(a) The even periodic extension is given by

$$f_{\text{even}}(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{\pi}{2}, \\ -1 & \text{for } \frac{\pi}{2} < t \le \pi, \\ 1 & \text{for } -\frac{\pi}{2} \le t < 0, \\ -1 & \text{for } -\pi < t < -\frac{\pi}{2}, \end{cases}$$

$$f_{\text{even}}(t + 2\pi) = f_{\text{even}}(t),$$

and is drawn below.



The fundamental period of this even extension is $\tau = 2\pi$, and its formula can be simplified to

$$f_{\text{even}}(t) = \begin{cases} 1 & \text{for } -\frac{1}{2}\pi \le t \le \frac{1}{2}\pi, \\ -1 & \text{for } \frac{1}{2}\pi < t < \frac{3}{2}\pi, \end{cases}$$

$$f_{\text{even}}(t + 2\pi) = f_{\text{even}}(t).$$

(b) Because the function $f_{\text{even}}(t)$ is even and has period $\tau = 2\pi$, its Fourier series takes the form

$$F_{\text{even}}(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nt).$$

The coefficient A_0 is given by

$$A_0 = \frac{2}{2\pi}\int_0^{\pi} f_{\text{even}}(t)\,dt = \frac{1}{\pi}\left(\int_0^{\pi/2} 1\,dt + \int_{\pi/2}^{\pi} (-1)\,dt\right) = 0.$$

The average value of f_{even} over its period is equal to zero.

The coefficients A_n are given by

$$\begin{aligned}A_n &= \frac{4}{2\pi}\int_0^{\pi} f_{\text{even}}(t)\cos(nt)\,dt \\ &= \frac{2}{\pi}\left(\int_0^{\pi/2}\cos(nt)\,dt - \int_{\pi/2}^{\pi}\cos(nt)\,dt\right) \\ &= \frac{2}{n\pi}\left(\Big[\sin(nt)\Big]_0^{\pi/2} - \Big[\sin(nt)\Big]_{\pi/2}^{\pi}\right) = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right).\end{aligned}$$

When n is an even integer, $\sin(n\pi/2) = 0$. When n is an odd integer, we can put $n = 2m - 1$ for $m = 1, 2, 3, \ldots$ and use equation (24) to get

$$\sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{(2m-1)\pi}{2}\right) = (-1)^{m+1}.$$

Hence

$$F_{\text{even}}(t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}\cos((2m-1)t).$$

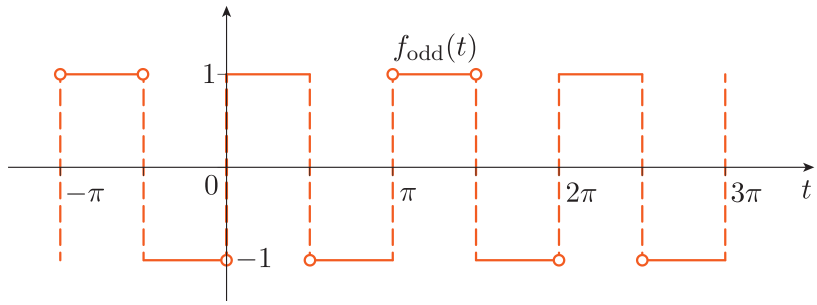
(c) The odd periodic extension is defined by

$$f_{\text{odd}}(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{\pi}{2}, \\ -1 & \text{for } \frac{\pi}{2} < t \le \pi, \\ -1 & \text{for } -\frac{\pi}{2} \le t < 0, \\ 1 & \text{for } -\pi < t < -\frac{\pi}{2}, \end{cases}$$

$$f_{\text{odd}}(t + 2\pi) = f_{\text{odd}}(t),$$

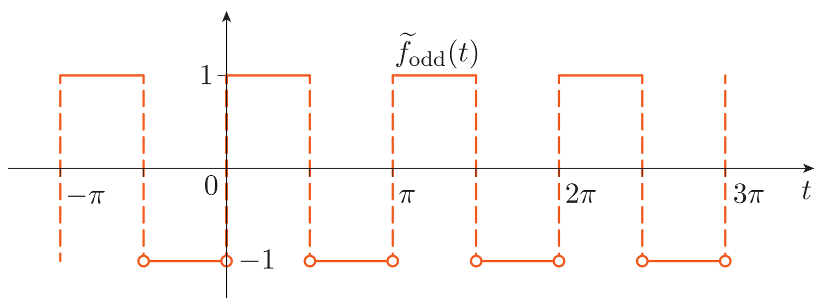
and is drawn below.

Do not worry too much about the precise placement of the open circles; you would not be penalised in assessments or the exam for getting this wrong.



Because of the way the discontinuities of this function are defined, it has fundamental period $\tau = 2\pi$.

(d) The graph of the odd extension is very similar to the following graph, except for the points of discontinuity.



This function has fundamental period π and is defined by

$$\widetilde{f}_{\text{odd}}(t) = \begin{cases} -1 & \text{for } -\frac{\pi}{2} < t < 0, \\ 1 & \text{for } 0 \leq t \leq \frac{\pi}{2}, \end{cases}$$

$$\widetilde{f}_{\text{odd}}(t+\pi) = \widetilde{f}_{\text{odd}}(t).$$

The Fourier series for this function was given in equation (27):

$$\widetilde{F}_{\text{odd}}(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(2(2n-1)t\right).$$

Since Fourier series cannot distinguish between functions that differ at isolated points, this is also the required Fourier series for the odd extension $f_{\text{odd}}(t)$.

Solution to Exercise 19

(a) $f(t)$ is discontinuous at $t=1$. Just below $t=1$, $f(t)=1$, and just above $t=1$, $f(t)=0$. Hence at $t=1$ the Fourier series converges to

$$F(1) = \tfrac{1}{2}.$$

(b) We can compare this with the Fourier series derived in Example 8:

$$F(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}\left((-1)^n - 1\right)\cos(n\pi t) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi}\sin(n\pi t).$$

But at $t=1$, $\sin(n\pi t)=0$ and $\cos(n\pi t) = (-1)^n$ for any integer n. So

$$F(1) = \frac{1}{2} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}((-1)^n - 1)(-1)^n.$$

Since $(-1)^{2n} = 1$, this rearranges to

$$\frac{1}{4} = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}(1-(-1)^n),$$

so

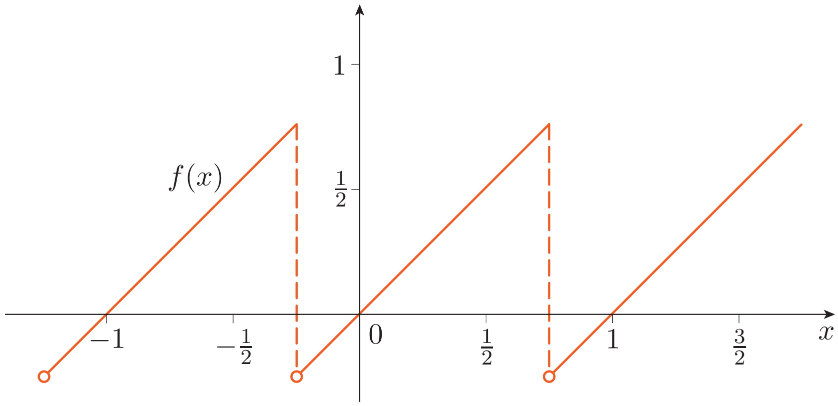
$$\begin{aligned} \frac{\pi^2}{4} &= \sum_{n=1}^{\infty} \frac{1}{n^2}(1-(-1)^n) \\ &= \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2}. \end{aligned}$$

Hence

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Solution to Exercise 20

The function $f(x)$ is sketched below.



This function has fundamental period $\tau = 1$ and is neither odd nor even, so its Fourier series takes the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\pi x) + \sum_{n}^{\infty} B_n \sin(2n\pi x).$$

If we were to take the fundamental interval of this function to be $-1/2 \le x \le 1/2$, it would be in two pieces and we would have to perform two integrals for each Fourier coefficient. Instead, let us use equations (41)–(43) with $t_0 = -\frac{1}{4}$ and $\tau = 1$. Then

$$\begin{aligned} A_0 &= \frac{1}{\tau}\int_{t_0}^{t_0+\tau} f(x)\,dx \\ &= \frac{1}{1}\int_{-1/4}^{3/4} x\,dx \\ &= \tfrac{1}{2}\left[x^2\right]_{-1/4}^{3/4} = \tfrac{1}{2}\left(\tfrac{9}{16} - \tfrac{1}{16}\right) = \tfrac{1}{4}. \end{aligned}$$

Similarly,

$$\begin{aligned} A_n &= \frac{2}{\tau}\int_{t_0}^{t_0+\tau} f(x)\cos\left(\frac{2n\pi x}{\tau}\right)dx \\ &= 2\int_{-1/4}^{3/4} x\cos(2n\pi x)\,dx. \end{aligned}$$

Using the standard integral in equation (33), we get

$$\begin{aligned} A_n &= \frac{2}{(2n\pi)^2}\left[\cos(2n\pi x) + 2n\pi x\sin(2n\pi x)\right]_{-1/4}^{3/4} \\ &= \frac{2}{(2n\pi)^2}\bigg(\cos\left(\tfrac{3}{2}n\pi\right) + \tfrac{3}{2}n\pi\sin\left(\tfrac{3}{2}n\pi\right) \\ &\qquad - \cos\left(-\tfrac{1}{2}n\pi\right) + \tfrac{1}{2}n\pi\sin\left(-\tfrac{1}{2}n\pi\right)\bigg). \end{aligned}$$

When n is any integer, we have

$$\cos\left(-\tfrac{1}{2}n\pi\right) = \cos\left(-\tfrac{1}{2}n\pi + 2n\pi\right) = \cos\left(\tfrac{3}{2}n\pi\right)$$

and

$$\sin\left(-\tfrac{1}{2}n\pi\right) = \sin\left(-\tfrac{1}{2}n\pi + 2n\pi\right) = \sin\left(\tfrac{3}{2}n\pi\right).$$

So

$$A_n = \frac{1}{n\pi}\sin\left(\tfrac{3}{2}n\pi\right) \quad (n = 1, 2, 3, \ldots).$$

Similarly,

$$\begin{aligned} B_n &= \frac{2}{\tau}\int_{t_0}^{t_0+\tau} f(x)\sin\left(\frac{2n\pi x}{\tau}\right)dx \\ &= 2\int_{-1/4}^{3/4} x\sin(2n\pi x)\,dx. \end{aligned}$$

Using the standard integral in equation (32), we get

$$\begin{aligned} B_n &= \frac{2}{(2n\pi)^2}\big[\sin(2n\pi x) - 2n\pi x\cos(2n\pi x)\big]_{-1/4}^{3/4} \\ &= \frac{2}{(2n\pi)^2}\Big(\sin\left(\tfrac{3}{2}n\pi\right) - \tfrac{3}{2}n\pi\cos\left(\tfrac{3}{2}n\pi\right) \\ &\qquad - \sin\left(-\tfrac{1}{2}n\pi\right) - \tfrac{1}{2}n\pi\cos\left(-\tfrac{1}{2}n\pi\right)\Big). \end{aligned}$$

We use $\cos\left(-\frac{1}{2}n\pi\right) = \cos\left(\frac{3}{2}n\pi\right)$ and $\sin\left(-\frac{1}{2}n\pi\right) = \sin\left(\frac{3}{2}n\pi\right)$ again to give

$$B_n = -\frac{1}{n\pi}\cos\left(\tfrac{3}{2}n\pi\right) \quad (n = 1, 2, 3, \ldots).$$

Hence the Fourier series is given by

$$F(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left(\sin\left(\tfrac{3}{2}n\pi\right)\cos\left(2n\pi x\right) - \cos\left(\tfrac{3}{2}n\pi\right)\sin\left(2n\pi x\right)\right).$$

Solution to Exercise 21

The function $c(t)$ is continuous, as can be seen from its graph, and $s'(t)$ is *minus* its derivative. The Fourier series for $s'(t)$ is denoted by $S(t)$ and is equal to *minus* the derivative of $C(t)$. Hence

$$\begin{aligned} S(t) &= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\frac{d}{dt}\big(\cos(2nt)\big) \\ &= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\big(-2n\sin(2nt)\big) \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{4n^2-1}\sin(2nt). \end{aligned}$$

Solution to Exercise 22

(a) $f(t)$ has fundamental period $\tau = 2$. Hence from equations (48) and (49), the exponential Fourier series is given by

$$F(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{in\pi t},$$

with Fourier coefficients

$$\begin{aligned}
C_n &= \frac{1}{2}\int_{-1}^{1} f(t)\, e^{-in\pi t}\, dt \\
&= \frac{1}{2}\int_{-1}^{1} e^{(\alpha - in\pi)t}\, dt \\
&= \frac{1}{2(\alpha - in\pi)}\left[e^{(\alpha - in\pi)t}\right]_{-1}^{1} \\
&= \frac{1}{2(\alpha - in\pi)}\left(e^{(\alpha - in\pi)} - e^{-(\alpha - in\pi)}\right).
\end{aligned}$$

We have

$$e^{(\alpha - in\pi)} = e^{\alpha} e^{-in\pi} = e^{\alpha}\left(\cos(n\pi) - i\sin(n\pi)\right) = (-1)^n\, e^{\alpha},$$

and similarly,

$$e^{-(\alpha - in\pi)} = (-1)^n\, e^{-\alpha}.$$

Hence

$$C_n = \frac{(-1)^n}{2(\alpha - in\pi)}\left(e^{\alpha} - e^{-\alpha}\right).$$

Multiplying by $1 = (\alpha + in\pi)/(\alpha + in\pi)$, we get

$$C_n = \frac{(-1)^n (e^{\alpha} - e^{-\alpha})}{2(\alpha^2 + n^2\pi^2)}\,(\alpha + in\pi).$$

The exponential Fourier series for $f(t)$ is therefore

$$F(t) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (e^{\alpha} - e^{-\alpha})}{2(\alpha^2 + n^2\pi^2)}\,(\alpha + in\pi)\, e^{in\pi t}.$$

(b) Using equations (51) and the expression for C_n calculated in part (a), we get

$$\begin{aligned}
A_0 &= C_0 = \frac{e^{\alpha} - e^{-\alpha}}{2\alpha}, \\
A_n &= 2\,\mathrm{Re}(C_n) = \frac{(-1)^n (e^{\alpha} - e^{-\alpha})\alpha}{\alpha^2 + n^2\pi^2}, \\
B_n &= -2\,\mathrm{Im}(C_n) = -\frac{(-1)^n (e^{\alpha} - e^{-\alpha})n\pi}{\alpha^2 + n^2\pi^2}.
\end{aligned}$$

The trigonometric Fourier series for $f(t)$ is therefore given by

$$F(t) = (e^{\alpha} - e^{-\alpha})\left[\frac{1}{2\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2 + n^2\pi^2}\left(\alpha\cos(n\pi t) - n\pi\sin(n\pi t)\right)\right].$$

Acknowledgements

Grateful acknowledgement is made to the following sources:

Figure 25: http://en.wikipedia.org/wiki/File:1st_commercial_Moog_synthesizer_(1964,_commissioned_by_the_Alwin_Nikolai_Dance_Theater_of_NY)_@_Stearns_Collection_(Stearns_2035),_University_of_Michigan.jpg. This file is licensed under the Creative Commons Attribution-NoDerivatives Licence http://creativecommons.org/licenses/by-nd/3.0.

Figure 42: Ukrainian Ministry of Emergencies.

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